Well-Quasi-Orders in Subclasses of Bounded Treewidth Graphs and their Algorithmic Applications

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Abstract. We show that three subclasses of bounded treewidth graphs are well-quasi-ordered by refinements of the minor order. Specifically, we prove that graphs with bounded vertex-covers are well quasi ordered by the induced subgraph order, graphs with bounded feedback-vertex-set are well quasi ordered by the topological-minor order, and graphs with bounded circumference are well quasi ordered by the induced-minor order. Our results give algorithms for recognizing any graph family in these classes which is closed under the corresponding minor order refinement.

1 Introduction

Treewidth is a graph invariant which is central in the design of algorithms for NP-hard graph problems. It plays a prominent role in the design of both approximation algorithms (see *e.g.* [9, 19]), and parameterized algorithms [11, 18, 37]. Its importance rests on the fact that many natural problems turn out to be fixed-parameter tractable (FPT) when instance treewidth is taken to be the parameter. There are currently several general methods for classifying problems as FPT when the parameterization includes treewidth, such as tree-decomposition dynamic-programming [3, 4, 6] and Courcelle's Theorem [7].

Despite the general success of treewidth as a parameter that leads to FPT algorithms, there are still important problems that have no known FPT algorithm when treewidth is taken as the parameter (see *e.g.* [14]). It is therefore reasonable to explore the development of general methods for restricted subclasses of bounded treewidth graphs, such as graphs of bounded vertex cover number. In this paper we explore well quasi ordering as a means towards this aim.

Using well quasi ordering to derive algorithmic results is inspired by the graph minor project of Robertson and Seymour. This project gave rise to the most striking results currently known in algorithmic graph-theory, yielding polynomial-time algorithms for all types of surface embedding problems [40], topological problems of different kinds [15], and various types of problems that emerge in VLSI design [16]. The *existence* of all of these remarkable algorithms follows as an easy corollary from the following two facts:

1. There is no infinite antichain of finite graphs in the minor order. More precisely, finite graphs are *well-quasi-ordered* under the minor order [41].
2. There is an efficient *order testing* algorithm for the minor order. Specifically, one can test whether a given graph on k vertices is a minor of given graph on n vertices in $f(k) \cdot n^3$ time [40].

A consequence of these two major results is that any minor ideal, that is, any set of graphs closed under minors, can be recognized in polynomial time. Since many natural graph problems turn out to be minor ideal recognition problems, the above two results provide a type of *meta-algorithm* for all such problems. In attempting to generalize this result, one can consider *refinements* of the minor order, that is, orders $\preceq$ for which H is a minor of G whenever $H \preceq G$, but not necessarily the other way around. Since the minor order is typically defined via graph operations, a natural way to obtain refinements of this order is by restricting these operations. Table 1 below gives well-known graph orders that can be obtained this way:

	vertex deletions	edge deletions	topological contractions	contractions
induced subgraph	$\surd$			
subgraph	$\surd$	$\surd$		
induced topological minor	$\surd$		$\surd$	
topological minor	$\surd$	$\surd$	$\surd$	
induced minor	$\surd$		$\surd$	$\surd$
minor	$\surd$	$\surd$	$\surd$	$\surd$

Table 1. Six graph orders defined by four different types of graph operations. The *vertex* and *edge deletion* operations are the self-explanatory operations resulting in the removal of a vertex or an edge from the graph, respectively. A *contraction* in a graph is the operation of replacing two adjacent vertices u and v in the graph by a vertex which is connected to the set of all neighbors of u and v. If either u or v have exactly two neighbors, then this operation is called a *topological contraction*. As an example, according to the table above, a graph H is an induced topological minor of a graph G if (an isomorphic copy of) H can be obtained by vertex deletions and topological contractions in G.

In contrast to the graph minor order, none of the other orders in Table 1 above defines a well-quasi-order on the set of all graphs, nor on the set of graphs of bounded treewidth [45]. Our main result shows that nevertheless these orders are in fact well-quasi-orders on some natural subclasses of bounded treewidth graphs. Recall that a vertex cover in a graph is a set of vertices which covers all edges in the graph, a feedback-vertex-set is a set of vertices which covers all cycles in the graph, and the circumference of a graph is the maximum length of any of its cycles. Bounding each of these graph invariants results in a bound on the treewidth as well, so graphs which have a bound on any of these parameters form a subclass of bounded treewidth graphs. The main result of this paper is the following theorem:

Theorem 1. *Let k be any fixed positive integer. Then the following are all well-quasi-ordered sets:*

- *The set of all graphs with vertex-cover at most k under the induced subgraph order.*
- *The set of all graphs with feedback-vertex-set at most k under the topological-minor order.*
- *The set of all graphs with circumference at most k under the induced-minor order.*

We remark that the first item of Theorem 1 above is a special case of Ding's Bounded Paths Theorem [10] (see Section 3), while the second item was proved already by Mader in the early 1970's [29]. However, we provide here a single proof for both of these facts which is considerably simpler than the original proofs of Ding and Mader. Furthermore, our proof is given in a general context, and can be applied for showing the well-quasi-orderness of other classes graphs (see Remark 3 in Section 4). The last item of Theorem 1 is new to the best of our knowledge.

In the context of fixed-parameter algorithms, Theorem 1 has interesting applications which are encapsulated in the corollary below (see Section 2 for details):

Corollary 1. *Let k be some fixed positive integer. There is a linear-time algorithm for recognizing:*

- *Any family of graphs with vertex-cover at most k that is closed under subgraphs.*
- *Any family of graphs with feedback-vertex-set at most k that is closed under topological minors.*
- *Any family of graphs with circumference at most k that is closed under induced-minors.*

1.1 Organization

The remainder of this paper is devoted to proving Theorem 1. In Section 2 we briefly review some basic terminology and notation, and discuss the fundamentals behind the method of well-quasi-ordering. Section 3 then provides the general framework for proving Theorem 1 by introducing what we call well quasi order identification tools. The correctness of these tools is proved in Sections 4 and 5. Section 6 discusses three concrete applications of Corollary 1, and in Section 7 we summarize the paper and discuss some possible directions for future work.

1.2 Related work

There has been considerable amount of work for showing that certain graph classes are well quasi ordered by various graph orders. By far the most well-known of these results is the Graph Minor Theorem due to Robertson and Seymour [41]. Kruskal [27] proved that trees are well quasi ordered by the topological minor order. Thomas showed that graphs of treewidth at most 2 are well quasi ordered by induced minors [45]. Damaschke showed that the class of cographs are well quasi ordered by induced subgraphs [8], and Ding showed that this is true also for graphs with no paths of a certain fixed length [10]. Moreover, Ding gave a characterization of all subgraph ideals that are well quasi ordered by either the subgraph or induced subgraph orders. Finally, Petkovšek showed that the class of k-letter graphs are well quasi ordered by induced subgraphs, for every fixed $k \in \mathbb{N}$ [39].

Our work is very much related to the study of parameterized problems using alternative, also known as the *complexity ecology of parameters* [13, 14, 36]. Some examples of results in this direction include FPT algorithms for graph layout problems parameterized by the vertex-cover [17], an algorithm for Graph Isomorphism parameterized by feedback-vertex-set [26], and a polynomial kernel for the Vertex Cover problem parameterized by feedback-vertex-set [22]. In this context, it is also worth mentioning the recent result of Lampis [28] concerning algorithmic meta-theorems for model checking problems parameterized by either vertex-cover or feedback-vertex-set. The results presented in this paper can also be thought as algorithmic meta-theorems, albeit of a different kind than those in [28].

2 Preliminaries

In the following section we discuss basic notation used throughout the paper, and also review the fundamentals behind the method of well quasi ordering. For a graph G we let $V(G)$ denote its vertex-set, and $E(G)$ denote its edge set. Given a subset of vertices $A \subseteq V(G)$, we let $G - A$ denote the graph H with $V(H) := V(G) \setminus A$ and $E(H) := \{\{u,v\} \in E(G) : u, v, \in V(G) \setminus A\}$.

Next we define the graph orders that we will be working with in this paper. For convenience, we define these orders via mappings satisfying certain conditions. It is well known that this is equivalent to defining the orders using graph operations as suggested in Section 1. Let H and G be two given graphs:

– A *subgraph embedding* of H in G is an injection $f : V(H) \to V(G)$ with $\{u,v\} \in E(H) \Rightarrow \{f(u), f(v)\} \in E(G)$.
– A *topological embedding* of H in G is a mapping $f := f_V \cup f_E$, where f_V is an injective function $f_V : V(H) \to V(G)$, and f_E is a function $f_E : E(H) \to 2^{V(G)}$ such that for all $\{u,v\} \in E(H)$, $f_E(\{u,v\})$ is a path in G between $f_V(u)$ and $f_V(v)$, and for all $\{u,v\} \neq \{u',v'\} \in E(H)$, $f_E(\{u,v\})$ and $f_E(\{u',v'\})$ share no common internal vertices.
– A *minor embedding* of H in G is a injective mapping $f : V(H) \to 2^{V(G)}$ with $f(v)$ connected in G for all $v \in V(H)$, $f(u) \cap f(v) = \emptyset$ for all $u \neq v \in V(H)$, and $\{u,v\} \in E(H) \Rightarrow \exists x \in f(u)$ and $\exists y \in f(v)$ with $\{x,y\} \in E(G)$.

Any of the three above embeddings can be strengthen to an induced embedding which corresponds to the case where edge removals are not allowed:

– An *induced subgraph embedding* of H in G is a subgraph embedding f of H in G with $\{f(u), f(v)\} \in E(G) \Rightarrow \{u,v\} \in E(H)$.
– An *induced-topological embedding* of H in G is a topological embedding $f := f_V \cup f_E$ of H in G where $\{u,v\} \notin E(H) \Rightarrow \{f_V(u), f_V(v)\} \notin E(G)$, and $f_E(\{u,v\})$ is an induced path in G, for all $\{u,v\} \in E(H)$.
– An *induced-minor embedding* of H in G is a minor embedding f of H in G such that $\exists x \in f(u)$ and $\exists y \in f(v)$ with $\{x,y\} \in E(G) \Rightarrow \{u,v\} \in E(H)$.

We write $H \subseteq G$ (resp. $H \trianglelefteq G$, $H \sqsubseteq G$) if there exists a subgraph (resp. topological-minor, minor) embedding of H in G. We use the superscript i to indicate the fact that there exists an induced embedding of the corresponding type. Thus, $H \trianglelefteq^i G$ for instance denotes that there exists an induced topological embedding of H in G. Note that all relations defined by the embedding above define a *quasi-order* (*i.e.*, transitive and reflexive) on the set of all graphs. Also observe that

$$H \subseteq^i G \Rightarrow H \subseteq G \Rightarrow H \trianglelefteq G \Rightarrow H \sqsubseteq G,$$

and that

$$H \subseteq^i G \Rightarrow H \trianglelefteq^i G \Rightarrow H \sqsubseteq^i G \Rightarrow H \sqsubseteq G.$$

Given an order $\preceq \in \{\subseteq, \subseteq^i, \trianglelefteq, \trianglelefteq^i, \sqsubseteq, \sqsubseteq^i\}$, a pair (G_i, G_j) in a sequence $G_1, G_2, G_3 \cdots$ of graphs is called a *good pair* if $G_i \preceq G_j$ and $i < j$. A *good sequence* w.r.t. $\preceq$ is an infinite sequence of graphs with a good pair, and a *bad sequence* is an infinite sequence which is not good. A class of graphs $\mathcal{G}$ is *well-quasi-ordered (wqo)* by $\preceq$ if contains no bad sequences w.r.t. $\preceq$. Since there are no infinite strictly descending sequences in $\mathcal{G}$ w.r.t. $\preceq$, the fact that $\mathcal{G}$ is wqo by $\preceq$ is equivalent to the fact that $\mathcal{G}$ contains no infinite anti-chains with respect to $\preceq$.

An *ideal* $\mathcal{I}$ in a wqo set $\langle \mathcal{G}, \preceq \rangle$ is a subset of $\mathcal{G}$ that is closed under $\preceq$. That is, $\mathcal{I}$ is an ideal of $\langle \mathcal{G}, \preceq \rangle$ iff $\mathcal{I} \subseteq \mathcal{G}$, and $H \in \mathcal{I}$ whenever $H \preceq G$ for some $G \in \mathcal{I}$. Ideals of wqo sets $\langle \mathcal{G}, \preceq \rangle$ have a property which is very interesting in our context: Consider the set

$$\mathrm{Forb}(\mathcal{I}) := \{H \in \mathcal{G} \setminus \mathcal{I} \, : \, H' \not\prec H \text{ for all } H' \in \mathcal{G} \setminus \mathcal{I}\}.$$

This set has the property that $G \in \mathcal{I}$ iff $H \not\preceq G$ for all $H \in \mathrm{Forb}(\mathcal{I})$, and thus it is called a *forbidden characterization* of $\mathcal{I}$. Furthermore, since $\preceq$ is wqo, $\mathrm{Forb}(\mathcal{I})$ is necessarily finite as it constitutes an anti-chain w.r.t $\preceq$. Thus, every ideal of a wqo set has a finite forbidden characterization. This implies that the $\mathcal{I}$-Recognition problem, the problem of determining whether a give graph G is in $\mathcal{I}$, can be solved efficiently whenever $(H, \preceq)$-Testing, the the problem of determining whether $H \preceq G$ for a given graph G, can be solved efficiently for any $H \in \mathcal{G}$.

Lemma 1 (Ideal Recognition Lemma). *Let $\langle \mathcal{G}, \preceq \rangle$ be a wqo set. Suppose that for any $H \in \mathcal{G}$, the $(H, \preceq)$-*Testing *problem can be solved in $O(n^c)$ for some constant $c \in \mathbb{N}$. Then for any ideal $\mathcal{I}$ in $\langle \mathcal{G}, \preceq \rangle$, the $\mathcal{I}$-*Recognition *problem can be solved in $O(n^c)$ time.*

Proof. The algorithm for $\mathcal{I}$-Recognition has all graphs in Forb($\mathcal{I}$) "hardwired" into it, and on input $G \in \mathcal{G}$, it simply checks whether $H \preceq G$ for any $H \in$ Forb($\mathcal{I}$), and determines that $G \notin \mathcal{I}$ iff any of these checks turns out positive. Correctness of this algorithm follows from the fact that Forb($\mathcal{I}$) is a forbidden characterization of $\mathcal{I}$. Furthermore, its running-time can be bounded by $O(n^c)$, $n := |V(G)|$, since the number and sizes of the elements in Forb($\mathcal{I}$) depends only on $\mathcal{I}$, and is constant with respect to n. □

Since all graph orders considered can be linear-time testable in bounded treewidth graphs by applying Courcelle's Theorem (see *e.g.* [2], Corollary 1 follows immediately from Theorem 1 by applying the Ideal Recognition Lemma.

3 Two Tools for Identifying WQOs

In this section we develop two tools that will help us in proving Theorem 1. These tools allow us to reduce the question of whether a given graph family is wqo by a particular order, to the question of whether a simpler family of labeled graphs is wqo by the natural extension of the order to labeled graphs.

We begin with some terminology. A *labeling* of a set of graphs $\mathcal{G}$ is a set of functions $\{\sigma_G : G \in \mathcal{G}\}$, where each σ_G is a mapping of the vertices of G to a set of labels Σ_G, *i.e.* $\sigma_G : V(G) \to \Sigma_G$. The set $\Sigma = \bigcup_{G \in \mathcal{G}} \Sigma_G$ is the set of labels assigned by σ to $\mathcal{G}$. We will often consider set of labels Σ that have an order $\preceq$ associated with them. If Σ is wqo by $\preceq$, we say that σ is a *wqo labeling w.r.t* $\preceq$. Well-quasi-ordered labelings of $\mathcal{G}$ allow us to refine our graph orders: Given a wqo labeling $\sigma = \{\sigma_G : G \in \mathcal{G}\}$ w.r.t $\preceq$, and a pair of graphs $H, G \in \mathcal{G}$, we will write

- $H \subseteq_\sigma G$ ($H \subseteq^i_\sigma G$) to denote that there is a (induced) subgraph embedding f of H in G with $\sigma_H(v) \preceq \sigma_G(f(v))$ for all $v \in V(H)$.
- $H \trianglelefteq_\sigma G$ ($H \trianglelefteq^i_\sigma G$) to denote that there is a (induced) topological embedding $f := f_V \cup f_E$ of H in G with $\sigma_H(v) \preceq \sigma_G(f_V(v))$ for all $v \in V(H)$.
- $H \sqsubseteq_\sigma G$ ($H \sqsubseteq^i_\sigma G$) to denote that there is a (induced) minor embedding f of H in G where for each $v \in V(H)$ there is some $x \in f(v)$ with $\sigma_H(v) \preceq \sigma_G(x)$.

Let us next give two important examples of well-quasi-ordered labeled graph families. The first is due to Ding regarding graphs with no paths of a certain fixed length, and the second is Kruskal's famous Tree Theorem:

Theorem 2 (Ding's Bounded Paths Theorem [10]). *For any $k \in \mathbb{N}$, the universe of all graphs with no paths of length greater than k is wqo by $\subseteq^i_\sigma$, for any wqo labeling σ.*

Theorem 3 (Kruskal's Labeled Trees Theorem [27]). *The set of all trees is wqo by $\trianglelefteq_\sigma$ for any wqo labeling σ.*

We next introduce another tool that we will use known in the literature as Higman's Lemma. Let $\preceq$ be a quasi-order on a set X, and let $[X]^{<\aleph_0}$ be the set of all finite subsets of X. For two subsets $A, B \in [X]^{<\aleph_0}$, we write $A \preceq B$ if there exists an injective mapping $f : A \to B$ such that $a \leq f(a)$ for all $a \in A$. Higman's Lemma is stated as following.

Lemma 2 (Higman's Lemma [21]). *If $\langle X, \leq \rangle$ is a wqo set, then so is $\langle [X]^{<\aleph_0}, \leq \rangle$.*

The following are two corollaries of the two theorems above. The first of these can also be derived from a classical result due to Higman, known as Higman's Lemma [21].

Corollary 2. *Let $\mathcal{I}$ be an induced subgraph ideal, and let σ be a wqo labeling. If the subset of all connected graphs in $\mathcal{I}$ is wqo by $\subseteq^i_\sigma$, then $\mathcal{I}$ is also wqo by $\subseteq^i_\sigma$.*

Proof. Let $\mathcal{I}'$ denote the set of all connected graphs in $\mathcal{I}$. Since $\mathcal{I}$ is an induced subgraph ideal, we have $\mathcal{I}' \subseteq \mathcal{I}$, and so $\langle \mathcal{I}', \subseteq^i_\sigma \rangle$ is a wqo set. The proof now follows directly by applying Higman's Lemma. □

Corollary 3. *The set of all graphs with no edges is wqo by $\subseteq^i_\sigma$ for any wqo labeling σ.*

Proof. Follows directly from Corollary 2. □

Corollary 4. *The set of all forests is wqo by $\trianglelefteq^i_\sigma$ for any wqo labeling σ.*

Proof. Observe that a tree H is a topological minor of another tree G iff H is an induced-topological minor of G. The statement above follows then by applying Corollary 2. □

We are now in position to describe our first wqo identification tool. This tool is especially suited for universes consisting of graphs which have a small subset of vertices whose removal leaves a very simple structured graph, *e.g.* graphs with bounded vertex-cover or bounded feedback-vertex-set. Given a set of graphs $\mathcal{G}$, and a natural k, let us denote by $\mathcal{G} + k$ the set of all graphs G that have a set of k vertices whose deletion results in a graph in $\mathcal{G}$.

Theorem 4 (WQO Identification Tool 1). *If an induced-subgraph ideal $\mathcal{I}$ is wqo by $\trianglelefteq_\sigma$ (resp. $\subseteq^i_\sigma$, $\subseteq_\sigma$) for any wqo labeling σ, then $\mathcal{I} + k$ is wqo under $\trianglelefteq$ (resp. $\subseteq$, $\subseteq^i$).*

Our second identification tool is concerned with the induced-minor order and 2-connected graphs. In general, a connected graph G is called *2-connected* if it has at least two vertices, and no removal of less than two vertices leaves G disconnected (the empty graph is assumed to be disconnected). The second identification tool deals with induced-subgraph ideals which have 2-connected graphs with very simple structure:

Theorem 5 (WQO Identification Tool 2). *If the subset of all 2-connected graphs in some induced-subgraph ideal $\mathcal{I}$ is wqo by $\sqsubseteq^i_\sigma$ for any wqo labeling σ, then $\mathcal{I}$ is wqo by $\sqsubseteq^i$.*

The next two sections are devoted each to proving Theorem 4 and Theorem 5. But for now, let us next show that our two identification tools imply Theorem 1.

Proof (of Theorem 1 assuming Theorem 4 and Theorem 5). We prove the first two items of the lemma using Theorem 4, and the last item using Theorem 5:

– Let $\mathcal{I}$ denote the induced-subgraph ideal consisting of all graphs with no edges. According to Corollary 3, we know that $\mathcal{I}$ is wqo by $\subseteq_\sigma$ for any wqo labeling σ. Furthermore, for any $k \in \mathbb{N}$, $\mathcal{I} + k$ is exactly the set of all graphs with vertex-cover at most k by definition. Thus, applying Theorem 4 we get that graphs with vertex-cover at most k are wqo by induced-subgraphs.

– For graphs with bounded feedback-vertex-set the argument is similar to the above. Note that if $\mathcal{I}$ is the induced-subgraph ideal of all forests, then for any $k \in \mathbb{N}$, $\mathcal{I} + k$ is precisely the set of all graphs with feedback-vertex-set at most k. Combining Corollary 4 and Theorem 4 therefore shows that graphs with feedback-vertex-set at most k are wqo by topological-minors.

– Let $\mathcal{I}$ denote the induced-subgraph ideal of all graphs with circumference at most k, and let $\mathcal{I}'$ denote the subset of 2-connected graphs in $\mathcal{I}$. Since any two vertices in a 2-connected graph are connected by at least two paths (according to Menger's Theorem [32]), and thus belong together to some cycle, we get that graphs in $\mathcal{I}'$ have no paths of length greater than k, due to the bound on the circumference of graphs in $\mathcal{I}$. Therefore, according to Ding's Theorem, we get that $\mathcal{I}'$ is wqo by $\subseteq^i_\sigma$ for any wqo labeling σ, and so it is also wqo by $\sqsubseteq^i_\sigma$ for any wqo labeling σ. Plugging this into Theorem 5 gives us that $\mathcal{I}$ is wqo by induced-minors.

□

4 Correctness of the First Tool

In this section we provide a proof for Theorem 4. We will specify only the proof for the $\trianglelefteq$ order, as the proof for the $\subseteq$ and $\subseteq^i$ orders follows the same lines. To start with, we will assume we have a positive integer k, and an ideal $\mathcal{I}$ which is wqo by $\trianglelefteq_\sigma$ for any wqo labeling σ. We will show that any infinite sequence of graphs in $\mathcal{I} + k$ is good, *i.e.* it has graph which is a topological minor of another graph succeeding it in the sequence.

Let $\{G_i\}_{i=1}^\infty$ be any infinite sequence in $\mathcal{I} + k$. By definition, each graph G_i in this sequence has a subset of k vertices U_i with $G_i - U_i \in \mathcal{I}$. Let V_i denote the subset of vertices $V(G_i) \setminus U_i$. We construct a labeling $\sigma = \{\sigma_i : i \in \mathbb{N}\}$ on $\{G_i : i \in \mathbb{N}\}$ in a way that codifies the adjacency of vertices in U_i with vertices of V_i, for each $i \in \mathbb{N}$. For this, σ_i first assigns each vertex $u \in U_i$ an arbitrary distinct label $\sigma_i(u) \in \{1, \ldots, k\}$, and then it assigns a label in $2^{\{1,\ldots,k\}}$ to each $v \in V_i$ by

$$\sigma_i(v) := \{\, x \;\; : \;\; \exists u \in U_i \text{ with } \{u, v\} \in E(G_i) \text{ and } \sigma_i(u) = x \,\}.$$

See Fig. 1 for an example. Observe that since the set of labels Σ assigned by σ is finite, it is wqo by equality, and σ is a wqo labeling on $\{G_i : i \in \mathbb{N}\}$ w.r.t. $=$.

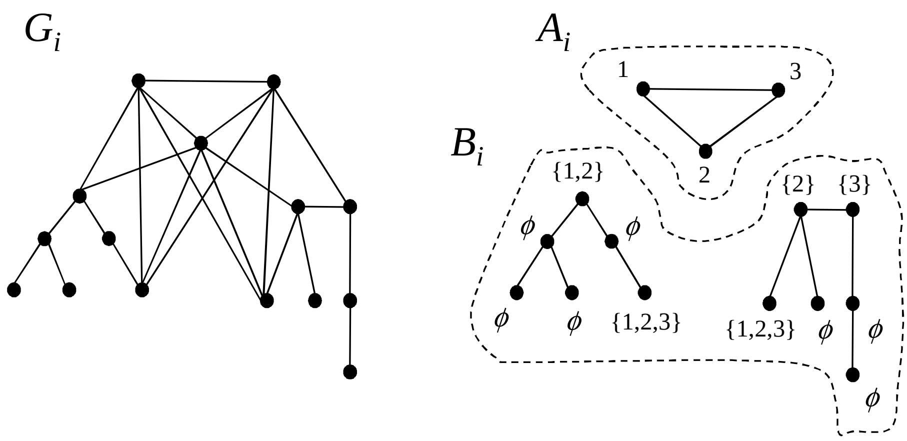

Fig. 1. The labeling used in proving the first identification tool.

Now, for each $i \in \mathbb{N}$, let A_i denote the graph $G_i - V_i$, and let B_i denote $G_i - U_i$. Then $B_i \in \mathcal{I}$ for all $i \in \mathbb{N}$. Since there are only finitely many graphs A_i under isomorphism, and only finitely many ways to label the vertices of these graphs with distinct labels in $\{1, \ldots, k\}$, there must be an infinite subsequence $G_{i_1}, G_{i_2}, \ldots$ in $\{G_i\}_{i=1}^{\infty}$ with $A_{i_1} \trianglelefteq_\sigma A_{i_2} \trianglelefteq_\sigma \cdots$. By our assumption, the family of graphs $\{B_{i_j} : j \in \mathbb{N}\} \subseteq \mathcal{I}$ is wqo by $\trianglelefteq_\sigma$, and so there must be a good pair (B_{i_x}, B_{i_y}) in $\{B_{i_j}\}_{j=1}^{\infty}$ with respect to $\trianglelefteq_\sigma$. Write $i = i_x$ and $j = i_y$. We argue that:

$$A_i \trianglelefteq_\sigma A_j \text{ and } B_i \trianglelefteq_\sigma B_j \text{ implies } G_i \trianglelefteq G_j$$

Let $g := g_V \cup g_E$ and $h := h_V \cup h_E$ denote the topological embedding of A_i in A_j and B_i in B_j, respectively. Also, let α_E denote the mapping that maps edge $\{u, v\} \in G_i$ with $u \in A_i$ and $v \in B_i$ to $\{g_V(u), h_V(v)\}$. Since $\sigma_i(u) = \sigma_j(g_V(u))$ and $\sigma_i(v) = \sigma_j(h_V(v))$, we have by definition of σ_i and σ_j:

$$\{u, v\} \in E(G_i) \Rightarrow \sigma_i(u) \in \sigma_i(v) \Rightarrow \sigma_j(g_V(u)) \in \sigma_j(h_V(v)) \Rightarrow \{g_V(u), h_V(v)\} \in E(G_j).$$

From this, it is immediate to verify that $f := f_V \cup f_E$ with $f_V := g_V \cup h_V$ and $f_E := g_E \cup h_E \cup \alpha_E$ is a topological embedding of G_i in G_j. It follows that $\{G_i\}_{i=1}^{\infty}$ is a good sequence, and as this sequence was chosen arbitrarily, this implies that $\mathcal{I} + k$ does not contain any bad sequences. This completes the proof of Theorem 4.

Remark 1. Theorem 4 does not hold for the induced-topological-minor order. Indeed the set of all wheels $\{W_i : i \in \mathbb{N}\}$, where W_i is obtained by adding to a cycle of length i a global vertex which is connected to all vertices on the cycle, forms an anti-chain w.r.t. $\trianglelefteq^i$, yet each W_i has a feedback-vertex-set of size two. The proof above breaks down when trying to show that $A_i \trianglelefteq_\sigma^i A_j$ and $B_i \trianglelefteq_\sigma^i B_j$ implies that $G_i \trianglelefteq^i G_j$, since it might be the case that edges from A_j to B_j must be deleted in order to properly map B_i to B_j in G_j.

Remark 2. There are induced subgraph ideals which are wqo by $\subseteq^i$ (resp. $\subseteq$) but are not wqo by $\subseteq_\sigma^i$ (resp. $\subseteq_\sigma$) for some wqo labeling σ. Consider, for instance, the induced subgraph ideal $\mathcal{I}$ of all graphs for which all connected components are simple paths. Let σ be a labeling assigning labels from the set $\Sigma := \{0, 1\}$. Then σ is a wqo labeling w.r.t. $=$. Now if σ assigns each endpoint of a path in a graph of $\mathcal{I}$ the label 1, and all remaining vertices are assigned the label 0, then it is not difficult to see that $\mathcal{I}$ is not wqo by $\subseteq_\sigma^i$.

Remark 3. Theorem 4 could be used to show that other classes of graphs not covered in Theorem 1 are wqo by one of the graph minor refinements. For instance, Damaschke showed that the class of cographs are wqo by induced subgraphs [8], and his proof can easily be adapted to the case where the graphs are labeled by a wqo labeling. Theorem 4 then shows that for each $k \in \mathbb{N}$, the class of graphs where one can delete k vertices in order to obtain a cograph is wqo by induced subgraphs. Note that this class of graphs includes the class of graphs with k-cluster-deletion number (*cf.* [20]). As another example, using the extended Graph Minor Theorem for graphs labeled by wqo labelings [42], and the fact that the minor and topological minor orders coincide for graphs of maximum degree 3, Theorem 4 yields that graphs with k vertices whose deletion results in a graph of maximum degree 3 are wqo by topological minors.

5 Correctness of the Second Tool

We next provide the proof for Theorem 5. We begin by introducing some additional terminology: A *rooted graph* is a pair (G, v) where G is a graph and v is a single distinguished vertex v of G

referred to as its *root*. Thus two rooted graphs with the same vertex and edge set, but with different roots, are considered different. Apart from the following definition, we will omit the parentheses notation and simply state that G is a rooted graph with $root(G) = v$.

Definition 1 (Rooted Closure). *The rooted closure of an induced subgraph ideal* $\mathcal{I}$*, denoted* $\mathcal{I}_r$*, is defined as the induced subgraph ideal of rooted graphs* $\mathcal{I}_r := \{(G, v) : G \in \mathcal{I}, v \in V(G)\}$.

We say that an induced-minor embedding f of a rooted graph H in a rooted graph G *preserves roots* if $root(G) \in f(root(H))$, and we will write $H \sqsubseteq^i G$ (and say that H is an induced minor of G) only when there exists a root-preserving minor-embedding of H in G. Our main interest in rooted graphs lies in the above refinement of minor embeddings, and in the obvious fact that $\mathcal{I}$ is wqo under $\sqsubseteq^i$ whenever $\mathcal{I}_r$ is wqo under $\sqsubseteq^i$.

Another important notion we need to introduce before beginning the proof of Theorem 5 is the notion of minimal bad sequences, a concept first introduced by Nash-Williams [35], and later also used by Kruskal in proving his Labeled Forests Theorem:

Definition 2 (Minimal Bad Sequence). *A bad sequence* $G_1, G_2, \ldots$ *is* minimal *if for every bad sequence* $H_1, H_2, \ldots$*, whenever* $|V(H_j)| < |V(G_j)|$ *for some* j*, there is always some* $i < j$ *such that* $|V(G_i)| < |V(H_i)|$.

Let $\mathcal{I}$ be an induced subgraph ideal whose subset of 2-connected graphs are wqo by $\sqsubseteq^i_\sigma$, for any wqo labeling σ. According to Corollary 2, we can assume that all graphs in $\mathcal{I}$ are connected. To prove the theorem, we assume that $\mathcal{I}$ is not wqo by $\sqsubseteq^i$, which implies that $\mathcal{I}_r$ is also not wqo by $\sqsubseteq$, and arrive at a contradiction by showing that in $\mathcal{I}_r$ contains no bad sequences.

Consider a minimal bad sequence in $\mathcal{I}_r$. According to Kruskal's Labeled Trees Theorem, this sequence contains an finite number of trees, and so let $G_1, G_2, \ldots$ denote the sequence obtained after removing all forests from the original sequence. A *block* in a graph G_i, $i \in \mathbb{N}$, is a maximal 2-connected induced subgraph of G. For each $i \in \mathbb{N}$, select a block A_i in G_i which contains $root(G_i)$, and let C_i denote the set of cutvertices of G_i that are included in A_i. For each cutvertex $c \in C_i$, let B^i_c denote the connected component in $G_i - (V(A_i) \setminus C_i)$ including the vertex c and made into a rooted graph by setting $root(B^i_c) = c$ (see Fig. 2). Observe that for any $c \in C_i$, we have $B^i_c \sqsubseteq^i G_i$ by the induced-minor root-preserving embedding f that maps every non-root vertex $v \neq c$ of B^i_c to itself, and has $f(c) = A_i \ni root(G_i)$. We argue that:

$$\text{The family of rooted graphs } \mathcal{B} = \{B^i_c : c \in C_i,\, i \in \mathbb{N}\} \text{ is wqo by } \sqsubseteq^i .$$

To see this, let $\{H_j\}_{j=1}^\infty$ be any sequence in $\mathcal{B}$, and for every $j \in \mathbb{N}$, choose an $i(j)$ for which $H_j = B^i_c$ for some $c \in C_i$. Pick a j with smallest $i(j)$, and consider the sequence

$$G_1, \ldots, G_{i(j)-1}, H_j, H_{j+1}, \ldots$$

Then this sequence is good by the minimality of $\{G_i\}_{i=1}^\infty$, and by our selection of j, and so it contains a good pair (G, G'). Now, G cannot be among the first $i(j) - 1$ elements of this sequence, since otherwise $G' = H_{j'}$ for some $j' \geq j$, and we will have

$$G \sqsubseteq^i G' = H_{j'} = B^{i(j')}_c \sqsubseteq^i G_{i(j')},$$

implying that $(G, G_{i(j')})$ is a good pair in the bad sequence $\{G_i\}_{i=1}^\infty$. Thus, (G, G') must be a good pair in $\{H_j\}_{j=1}^\infty$, and so $\{H_j\}_{j=1}^\infty$ is good.

We next use the above to show that $\{G_i\}_{i=1}^\infty$ has a good pair, bringing us to our desired contradiction. For this, we will label the graph family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ so that each cutvertex c of

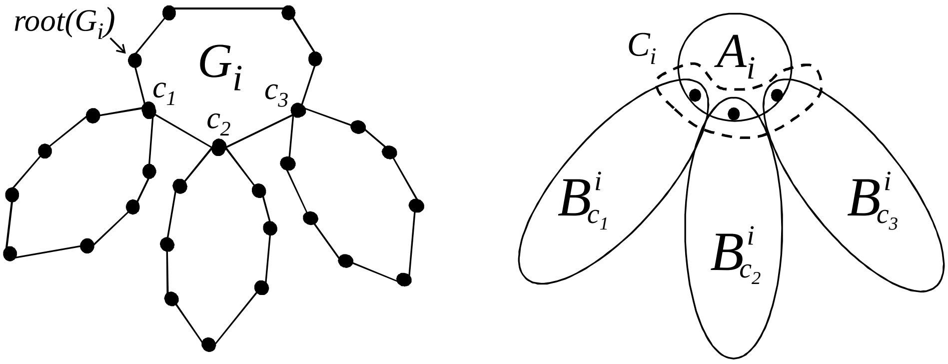

Fig. 2. The notation used in proving the second identification tool.

a graph A_i gets labeled by their corresponding connected component B^i_c of G_i, and the roots are preserved under this labeling. More precisely, for each A_i we define a labeling σ_i that assigns a pair of labels $(\sigma_i^{(1)}(v), \sigma_i^{(2)}(v))$ to every vertex $v \in V(G_i)$, where the labelings $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are defined by:

– $\sigma_i^{(1)}(v) = 1$ if $v = root(G_i)$, and otherwise $\sigma^{(1)}(v) = 0$.
– $\sigma_i^{(2)}(v) = B^i_v$ if $v \in C_i$, and otherwise $\sigma_i^{(2)}(v) = 0$.

The labeling σ of $\mathcal{A}$ is then $\{\sigma_i : i \in \mathbb{N}\}$. We define a quasi-ordering $\preceq$ on the set of labels Σ assigned by σ. For two labels $(\varsigma_a^{(1)}, \varsigma_a^{(2)}), (\varsigma_b^{(1)}, \varsigma_b^{(2)}) \in \Sigma$, we define

$$(\varsigma_a^{(1)}, \varsigma_a^{(2)}) \preceq (\varsigma_b^{(1)}, \varsigma_b^{(2)}) \iff \varsigma_a^{(1)} = \varsigma_b^{(1)} \text{ and } \varsigma_a^{(2)} \sqsubseteq^i \varsigma_b^{(2)}.$$

Observe that the $\sqsubseteq^i$ order above is between rooted graphs. Also, we allow 0 to be $\sqsubseteq^i$-comparable only to itself. It is not difficult to see that since $\sqsubseteq^i$ is a wqo on $\mathcal{B}$, the $\preceq$ order is wqo on Σ. Thus, σ is wqo labeling on $\mathcal{A}$ w.r.t. σ. By the assumptions in the theorem, we know that $\mathcal{A}$ is wqo by $\sqsubseteq^i_\sigma$. It follows that there is a pair of graphs $A_i, A_j \in \mathcal{A}$ with $A_i \sqsubseteq^i_\sigma A_j$. To complete the proof we will show that:

$$A_i \sqsubseteq^i_\sigma A_j \Rightarrow G_i \sqsubseteq^i G_j.$$

To see this, let f be the induced-minor embedding of A_i in A_j. Then for each cutvertex $c \in C_i$, $f(c)$ contains a vertex $d \in C_j$ with $B^i_c \sqsubseteq^i B^j_d$. Let f_c denote the induced-minor root-preserving embedding of B^i_c in B^j_d. We construct an embedding $g : V(G_i) \to 2^{V(G_j)}$ defined by

$$g(v) = \begin{cases} f(v) & : \quad v \text{ is a vertex of } A_i \text{ and } v \notin C_i, \\ f_c(v) & : \quad v \text{ is a vertex of } B^i_c \text{ and } v \neq c, \\ f(v) \cup f_v(v) & : \quad v \in C_i. \end{cases}$$

We argue that g is an induced minor embedding of G_i in G_j.

To see this, first note that by the definitions of f and f_c, we have $g(u) \cap g(v) = \emptyset$ for any pair of distinct vertices u and v in G_i. Moreover, for any edge $\{u, v\}$ of G_i there is a vertex $x \in g(u)$ and a vertex $y \in g(v)$ with $\{x, y\}$ and edge in G_j. Thus what remains to be shown is that $g(u)$ is connected in G_j for every vertex u of G_i. This is obviously true when $u \notin C_i$, again by the definitions of f and f_c. If $u \in C_i$, then $f(u)$ contains a vertex $v \in C_j$ for which $B^i_u \sqsubseteq^i B^j_v$, and

v is also contained in $f_v(v)$ since f_v preserves roots. Thus, $g(u)$ is connected also when $u \in C_i$. Noting also that the labeling σ ensures that $root(G_j) \in g(root(G_i))$, we establish that $G_i \sqsubseteq^i G_j$. This completes the proof of Theorem 5.

Remark 4. The reader should observe that graphs of bounded circumference are not wqo by topological minors. To see this, consider the infinite anti-chain $\{C_i^* : i \in \mathbb{N}, i \geq 3\}$, where C_i^* is the graph obtained by taking C_i, the cycle of length i, doubling each edge in the cycle, and then subdividing (the converse operation of topological contraction) all edges in the resulting graph.

6 Applications

We next illustrate how Theorem 1 can be applied to concrete problems. The aim of this section is just to give examples, and not to provide a complete set of problems solvable using Theorem 1.

We begin with recognition of geometric intersection graphs. For a class of geometric objects $\mathcal{O}$, an *intersection graph of* $\mathcal{O}$ is a graph G such that there exists an injective mapping $f : V(G) \to \mathcal{O}$ where for any pair of vertices $u, v \in V(G)$ we have $\{u, v\} \in E(G) \iff f(u) \cap f(v) \neq \emptyset$. Classical examples include the class of interval graphs when $\mathcal{O}$ is the set of all intervals of the real line, the class of box graphs when $\mathcal{O}$ is the set of all boxes in the plane, and the class of all unit-disc graphs when $\mathcal{O}$ is the set of all unit-discs in the plane.

For a fixed class $\mathcal{O}$, the $\mathcal{O}$-Graph Recognition problem asks to determine whether a given input graph G is an intersection graph of $\mathcal{O}$. Many examples of $\mathcal{O}$-Graph Recognition are known to be NP-has, *e.g.* Box-Graph Recognition [46] and Unit-Disc-Graph Recognition [5]. We are not aware of any restricted graph class where these are known to be polynomial-time solvable. However, observe that both disc graphs and rectangle graphs, as well as any other geometric intersection graph class, form an induced-subgraph ideal. Thus, by the first item of Corollary 1 we get:

Corollary 5. *Let $\mathcal{O}$ be any fixed class of geometric objects. For any $k \in \mathbb{N}$, there is an $f(k) \cdot n$ time algorithm which determines whether a given graph G with n vertices and vertex-cover at most k is an intersection graph of $\mathcal{O}$.*

A notoriously hard class of geometric intersection graphs to recognize is the class of string graphs. The String Graph Recognition problem was only recently shown to be decidable [38, 44], and later it was shown to be NP-complete by Schaefer, Sedgwick, and Štefankovic [43], after Jan Kratochvil [25] showed it is NP-hard. However, as observed by *e.g.* [31], the class of string graphs do not only form an induced subgraph ideal, they also form an induced minor ideal. Thus, despite the fact that they do not have a finite forbidden induced minor characterization in general [24], Thus, by the third item of Corollary 1 we get:

Corollary 6. *For any $k \in \mathbb{N}$, there is an $f(k) \cdot n$ time algorithm which determines whether a given graph G with n vertices and circumference at most k is a string graph.*

As a final example, we consider a well studied problem known as Topological Bandwidth [23, 30, 33]. The Topological Bandwidth problem is the problem of determining, given a graph G and an integer ℓ, whether there exists a graph H with $|V(H)| = |V(G)|$ and $G \trianglelefteq H$ such that H admits a vertex-ordering $\pi : V(H) \to \{1, \ldots, n\}$ with $|\pi(v) - \pi(u)| \leq \ell$ for every $\{u, v\} \in E(H)$. This problem is known to be NP-hard even for planar graphs of maximum degree 3 [34], and not known to be fixed-parameter tractable in bounded treewidth graphs. Nevertheless, it is easy to see that the set of graphs with topological bandwidth at most ℓ is a topological minor ideal for any fixed $\ell \in \mathbb{N}$, and so by the second item of Corollary 1 we get:

Corollary 7. *For any $k, \ell \in \mathbb{N}$, there is an $f(k+\ell) \cdot n$ time algorithm which determines whether a given graph G with n vertices and feedback vertex-set at most k, has topological bandwidth at most ℓ.*

7 Discussion

In this paper we showed that for any fixed $k \in \mathbb{N}$, the class of graphs with a vertex cover of size at most k is well quasi ordered by the induced subgraph order, the class of graphs with a feedback vertex set of size at most k is well quasi ordered by the topological-minor order, and the class of graphs with circumference at most k is well quasi ordered by the induced-minor order. Using Courcelle's Theorem, we showed that our results yield algorithms for ideal recognition problems. We remark that the use of Courcelle's Theorem can be replaced by faster dynamic programming algorithms given in [1, 12].

It would be interesting to find other subclasses of bounded treewidth graphs which are wqo by one of the minor order refinements. Is there an interesting subclass that is well quasi ordered by induced topological minors? A related order to the minor order that has not been discussed in this paper is the immersion order, introduced by Nash Williams [35]. Robertson and Seymour proved that all finite graphs are wqo also under the immersion order [42]. Are there any interesting classes of graphs that are well quasi ordered by induced immersions?

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