# Nearly Tight Approximability Results for Minimum Biclique Cover and Partition

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**Abstract.** In this paper, we consider the *minimum biclique cover* and *minimum biclique partition* problems on bipartite graphs. In the minimum biclique cover problem, we are given an input bipartite graph $G = (V, E)$, and our goal is to compute the minimum number of complete bipartite subgraphs that cover all edges of $G$. This problem, besides its correspondence to a well-studied notion of *bipartite dimension* in graph theory, has applications in many other research areas such as artificial intelligence, computer security, automata theory, and biology. Since it is NP-hard, past research has focused on approximation algorithms, fixed parameter tractability, and special graph classes that admit polynomial time exact algorithms. For the minimum biclique partition problem, we are interested in a biclique cover that covers each edge exactly once.

We revisit the problems from approximation algorithms' perspectives and give nearly tight lower and upper bound results. We first show that both problems are NP-hard to approximate to within a factor of $n^{1-\varepsilon}$ (where $n$ is the number of vertices in the input graph). Using a stronger complexity assumption, the hardness becomes $\tilde{\Omega}(n)$, where $\tilde{\Omega}(\cdot)$ hides lower order terms. Then we show that approximation factors of the form $n/(\log n)^{\gamma}$ for some $\gamma > 0$ can be obtained.

Our hardness results have many consequences: (i) $\tilde{\Omega}(n)$ hardnesses for computing the Boolean rank and non-negative integer rank of an $n$-by-$n$ matrix (ii) $\tilde{\Omega}(n)$ hardness for minimizing the number of states in a deterministic finite automaton (DFA), given an $n$-state DFA as input, and (iii) $\tilde{\Omega}(\sqrt{n})$ hardness for computing minimum NFA from a truth table of size $n$. These results settle some of the most basic problems in the area of regular language optimization.

## 1 Introduction

We study the problem of covering the edges of a graph by bipartite complete subgraphs (or bicliques). In this problem, we are given a graph $G = (V, E)$, and our

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objective is to compute a collection of complete bipartite subgraphs of $G$ that together cover all edges of $G$, while minimizing the number of such subgraphs. The problem is referred to as the minimum biclique cover problem (BicliqueCover) in the optimization literature and has many applications, as well as connections, to other areas of computer science, such as automata and language theory [15], computer security [8], bioinformatics [24], graph drawing [9], and artificial intelligence. Besides these applications, computing a biclique cover of a graph is equivalent to other important notions in mathematics: Given an $m$-by-$n$ matrix $M$ over a Boolean algebra. The *Boolean rank* of $M$ is the minimum $k$ for which there exist two matrices $(A)_{m\times k}$ and $(B)_{k\times n}$ such that $M = AB$. It has been shown that computing Boolean rank of a matrix is equivalent to computing the bipartite dimension of a bipartite graph (see [14]).

In most applications, one may assume that graph $G$ is bipartite. This problem has received a large amount of attention from a number of research groups. Since the problem is NP-hard, various approaches have been used in studying the problem: approximation algorithms [28,15], heuristics [8], fixed parameter tractability [25], and investigation of special graph classes that admit fast, polynomial-time algorithms [3,4,22,23].

Orlin showed that the problem is NP-hard, even on bipartite graphs [26]. Later Simon showed that the problem is also NP-hard to approximate [28]. Gruber and Holzer used the construction in [28] to show that the problem is $n^{1/3-\varepsilon}$ and $m^{1/5-\varepsilon}$ hard to approximate respectively. On an upper bound side, no non-trivial approximation algorithm has been proposed. The problem can be, however, solved efficiently in many cases. For instance, the fixed-parameter tractability result is known [25], implying that the problem can be solved in time $f(k)poly(n)$ provided that the biclique cover of size $k$ exists. Also, the problem is polynomial time solvable in several graph classes, such as domino-free graphs, C4-free graphs, and bipartite permutation graphs (see [3] and references therein).

A problem closely related to BicliqueCover (but perhaps receives less attention from researchers) is called BicliquePartition where our goal is to find a cover in which each edge is covered by exactly one biclique. In contrast to BicliqueCover, only APX-hardness result has been shown for this problem. This result relies on the equivalence of BicliquePartition and the normal set basis problem shown to be NP-hard by [18], and a reduction from vertex cover.

A further related problem, called maximum edge biclique problem (MaxBiclique), receives a lot of attention from approximation algorithms community. Dawande, Keskinocak and Tayur [21] showed that the weighted version of the MaxBiclique in bipartite graphs is $NP$-complete, but they were not able to show that the unweighted version is hard also. This was later accomplished by Peeters [27] who proved that MaxBiclique in bipartite graphs is $NP$-complete. In terms of approximation hardness, Feige [10] shows that the problem is hard to within a factor of $n^{\varepsilon}$ assuming average-case complexity hypothesis. Ambühl et al. prove the same result under a more standard assumption [1].

### 1.1 Our Contributions

Our main result is informally summarized in the following theorem.

**Theorem 1 (Informal).** BicliqueCover *and* BicliquePartition *on bipartite graphs are (almost) as hard to approximate as graph coloring.*

Combining this theorem with the hardness results for graph coloring [12,20,29] implies that these problems do not admit $n^{1-\varepsilon}$ and $m^{1/2-\varepsilon}$ approximation algorithm unless $\mathsf{P} = \mathsf{NP}$. With a stronger complexity assumption of $\mathsf{NP} \not\subseteq \mathsf{BPTIME}(2^{poly \log n})$, this gives a stronger hardness result of $\frac{n}{2^{\log^{7/8+\varepsilon} n}}$ and $\frac{\sqrt{m}}{2^{\log^{7/8+\varepsilon} m}}$ for any $\varepsilon > 0$. (For the purpose of deriving our corollaries, it is important to state the bounds in terms of both $m$ and $n$).

We immediately obtain the hardness of approximating the rank of a matrix through the connections shown in [14]. Also, Amilhastre et al. and Gruber and Holzer [2,15] discovered (nearly tight) connections between BicliqueCover, BicliquePartition, and several minimization problems for regular languages. Combining our result with theirs yields new hardness results (proofs will appear in the full version). We summarize the consequences of our theorem below.

**Corollary 1.** *Unless* $\mathsf{NP}$ *has bounded-error randomized quasi-polynomial time algorithm, for all* $\varepsilon > 0$*, it is hard to:*

- *Approximate the Boolean rank and non-negative integer rank of an* $n$*-by-*$n$ *matrix to within a factor of* $\frac{n}{2^{\log^{7/8+\varepsilon} n}}$.
- *Approximate the number of states of minimum NFA accepting a language* $L$*, specified by an input truth table of size* $N$*, to within a factor of* $\frac{\sqrt{N}}{2^{\log^{7/8+\varepsilon} N}}$.
- *Approximate the minimum number of states of the minimum DFA accepting a language* $L$*, specified by an input* $n$*-state DFA of size* $n$*, to within a factor of* $\frac{n}{2^{\log^{7/8+\varepsilon} n}}$.

All these results are essentially tight. These problems are some of the most basic problems in regular language minimization (see the survey by Holzer and Kutrib and references therein [17]). Prior to our results, similar hardness results require (much stronger) cryptographic assumptions [13]. We remark another interesting aspect of our results: It is noted in [17] that the lower bounds provided by biclique edge cover technique " *... are not always tight and can be arbitrarily worse ...*" Our results show that biclique cover techniques can in fact provide tight lower bounds for many problems listed in the survey, hence providing an evidence that biclique cover and partition capture the computational complexity of regular language minimization problems.

Our proof follows the framework of graph product techniques, as introduced and used succesfully by Chalermsook et al. [5,7,6]. Roughly speaking, this framework reduces the task of proving hardness of approximation to that of proving graph product inequalities. In our case, this amounts to bounding the quantity $bc(B[G \cdot H])$, by some slowly growing function of $bc(B[H])$ and $bc(B[G])$ where $bc(H)$ denotes the size of minimum biclique cover of $H$, "$\cdot$" is the lexicographic product of graphs, and $B[\cdot]$ is the *bipartite double cover* transformation respectively. The main idea of the proof is to use an optimal vertex coloring of $\bar{G}$ together with biclique covering of $B[H]$ to suggest the biclique cover of $B[G \cdot H]$. We note that, while we give lower bound results, the flavor of our proofs is rather algorithmic: It illustrates how one can algorithmically utilize the coloring of graph $\bar{G}$ in minimizing the biclique covers in $B[G^k]$.

Our hardness results rule out approximation ratios $n^\delta$ for any $\delta \in (0,1)$, so it is natural to aim at mildly sub-linear approximation factors, e.g., $\frac{n}{(\log n)^\gamma}$ for some $\gamma > 0$. We investigate this direction and obtain the following results.

**Theorem 2.** *There is an approximation algorithm for* BicliqueCover *that achieves an approximation ratio of*

$$O\left(\min\left\{n/\sqrt{\log(n)}, m(\log\log m)^2/(\log^3 m)\right\}\right).$$

We remark that the upper and lower bounds match up to lower-order factors (in terms of $n$). The second result relies on the idea that one can reduce BicliqueCover to MaxClique on the complement of the conflict graph.

Using a standard reduction, we furthermore obtain the following result.

**Corollary 2.** *There is no poly-time algorithm to approximate* MaxWeightedBiclique *within factors of $n^{1-\varepsilon}$ and $m^{1/2-\varepsilon}$, respectively, for all $\varepsilon > 0$ unless $P = NP$, or within a factor of $O\left(\frac{\min\{n,\sqrt{m}\}}{2^{\log^{7/8+\varepsilon} n}}\right)$ for any $\varepsilon > 0$ unless* $\mathsf{NP} \subseteq \mathsf{BPTIME}(2^{poly \log n})$. *This holds even when edge-weights are in $\{0,1\}$.*

## 2 Preliminaries

We start by a formal treatment of our problem. A *biclique* is denoted by $K_{a,b}$ which is a complete bipartite graph $(A, B, F)$ such that $|A| = a$ and $|B| = b$. Given a graph $G = (V, E)$, we say that $S \subseteq V$ is a *biclique subgraph* of $G$ if and only if the induced subgraph $G[S]$ is a biclique $K_{a,b}$ for some $a, b$.

A *biclique cover* of $G$ is a collection of vertices $S_1, \ldots, S_k$ such that each $S_i$ is a biclique subgraph of $G$ and each edge $e \in E(G)$ appears at least once in some $G[S_i]$. In such case, we say that a biclique cover of size $k$ exists for $G$. Let $bc(G)$ denote the minimum number $k$ for which a biclique cover of size $k$ exists for $G$. In BicliqueCover, our goal is to compute $bc(G)$ on an input graph $G$. A *biclique partition* of $G$ is a biclique cover such that, each edge is covered exactly once. It follows from the definition that $bc(G) \leq bp(G)$ for any graph $G$.

A *clique partition* of $G$ is a partition of vertices $V(G)$ into $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$ such that each induced subgraph $G[V_i]$ is a clique. The *clique partition number* of $G$, denoted by $cp(G)$, is the minimum number $k$ such that a clique partition of $V(G)$ into $k$ components exist. The clique partition problem (PartitionIntoCliques) asks for computing the value of $cp(G)$.

Given a graph $G$, let $\chi(G)$ be the *chromatic number* of $G$ which is the minimum number of colors $c$ such that there exists a proper $c$-coloring of $G$. Let

$\mathcal{I}_G$ be the set of all independent sets in $G$. A valid fractional $c$-coloring of $G$ is an assignment $\psi : \mathcal{I}_G \to [0,1]$ with the guarantees: (i) $\sum_{S:v\in S} \psi(S) \geq 1$ for all $v$ and (ii) $\sum_{S\in\mathcal{I}_G} \psi(S) \leq c$. A *fractional chromatic number* of $G$, $\chi_f(G)$, is the minimum $c$ such that there exists a valid fractional $c$-coloring for $G$.

Notice that for any graph $G$, we have $\chi(G) = cp(\bar{G})$. Similarly to the notion of fractional chromatic number, we may define *fractional clique partition number* $cp_f(G)$ as $\chi_f(\bar{G})$. This implies that $\frac{cp(G)}{\log|V(G)|} \leq cp_f(G) \leq cp(G)$.

Feige and Kilian [12] proved the NP-hardness of approximating $\chi(G)$. Since $\chi(G) = cp(\bar{G})$, the same hardness result holds for PartitionIntoCliques. Their result can be summarized formally below.

**Theorem 3 ([12,29]).** *Let $\varepsilon > 0$ be a constant. Given a graph $G = (V, E)$, it is NP-hard to approximate $cp(G)$ to within a factor of $|V(G)|^{1-\varepsilon}$.*

Assuming a stronger (but still standard) complexity theoretic assumption, Khot and Ponnuswami proved the following result [20].

**Theorem 4.** *Let $\varepsilon > 0$ be a constant. It is hard to approximate $cp(G)$ for a graph $G = (V, E)$ to within a factor of $\frac{|V(G)|}{2^{\log^{3/4+\varepsilon}|V(G)|}}$ unless* $\mathsf{NP} \subseteq \mathsf{BPTIME}(2^{poly \log n})$.

## 3 Hardness of Approximation

In this section, we prove our hardness results. We start by explaining graph product terminologies and tools in the next subsection.

### 3.1 Graph Products

Let $G$ and $H$ be any graphs. The lexicographic product of $G$ and $H$, i.e. $G\cdot H$, is defined as follows. The vertex set of $G\cdot H$ is $V(G\cdot H) = V(G)\times V(H)$ and the edge set is $E(G\cdot H) = \{(u,a)(v,b) : uv \in E(G)\} \cup \bigcup_{u\in V(G)} \{(u,a)(u,b) : ab \in E(H)\}$. For an integer $k$, the term $G^k$ denotes a $k$-fold lexicographic product of $G$, i.e. $G^k = G\cdot G \ldots \cdot G$ ($k$ times). The following inequality is a standard fact.

**Lemma 1.** *For any graphs $G$ and $H$, $\chi_f(G)\chi(H) \leq \chi(G\cdot H) \leq \chi(G)\chi(H)$*

We show that the clique partition number satisfies similar properties with respect to lexicographic products. The proof will appear in the full version.

**Lemma 2 (Multiplicativity of $cp$).** $cp_f(G)cp(H) \leq cp(G\cdot H) \leq cp(G)cp(H)$

### 3.2 Proof of the Hardness Result

We prove the following connection between PartitionIntoCliques and BicliqueCover, which will be used in deriving our hardness results.

**Theorem 5.** *Let $G$ be any graph and $k$ be an integer. There is an algorithm that runs in time $|V(G)|^{O(k)}$ and constructs a bipartite graph $H$ such that $|V(H)| = \Theta(|V(G)|^k)$ and*

$$\left(\frac{cp(G)}{\log|V(G)|}\right)^k \leq bc(H) \leq bp(H) \leq cp(G)^k|V(G)|^3$$

Before proving this theorem, we show how to use it to derive our hardness results.

**Corollary 3.** *Let $\varepsilon > 0$. It is NP-hard to approximate* BicliqueCover *and* BicliquePartition *within factors of $n^{1-\varepsilon}$ and $m^{1/2-\varepsilon}$. Moreover, there are no polynomial time approximation algorithms for both problems with a guarantee in $\frac{n}{2^{\log^{7/8+\varepsilon} n}}$ or $\frac{\sqrt{m}}{2^{\log^{7/8+\varepsilon} n}}$ unless $\mathsf{NP} \subseteq \mathsf{BPTIME}(2^{poly \log n})$.*

*Proof.* Our reduction combines the reduction that gives hardness result PartitionIntoCliques with Thm. 5. Let $\mathcal{A}_{clique}$ be the algorithm (i.e. reduction) that takes a SAT instance $\varphi$ and produces graph $G$, with the following properties:

– (Yes-Instance:) If $\varphi$ is satisfiable, then $cp(G) \leq c$
– (No-Instance:) If $\varphi$ is not satisfiable, then $cp(G) \geq s$.

Let $g = s/c$ be the gap (hardness factor) given by the reduction $\mathcal{A}_{clique}$. For instance, Thm. 3 gives such a reduction with $c = |V(G)|^{\varepsilon}$, $s = |V(G)|^{1-\varepsilon}$, $g = |V(G)|^{1-2\varepsilon}$, and $|V(G)| = |\varphi|^{O(1)}$. Our reduction $\mathcal{A}^k_{biclique}$ first runs the algorithm $\mathcal{A}_{clique}$ to get the instance $G$ and then apply Thm. 5 on graph $G$. The theorem outputs graph $H$ with $N = |V(H)| = \Theta(|V(G)|^k)$.

Now analyze the gap given by our reduction $\mathcal{A}^k_{biclique}$. Applying the lower bound of Thm. 5, for the No-Instance, we get $bc(H), bp(H) \geq \frac{s^k}{(\log|V(G)|)^k}$. For the Yes-Instance, we would get $bc(H), bp(H) \leq c^k|V(G)|^3$. So the gap between Yes-Instance and No-Instance of reduction $\mathcal{A}^k_{biclique}$ is

$$g' = \left(\frac{s}{c}\right)^k \frac{1}{|V(G)|^3(\log|V(G)|)^k} = \frac{g^k}{|V(G)|^3(\log|V(G)|)^k}$$

This gap holds for both BicliquePartition and BicliqueCover. Roughly speaking the gap between our Yes-Instance and No-Instance is $g' \approx g^k$. Now we plug in the appropriate values to obtain the desired hardness results.

If we start from Thm. 3, we have the starting hardness gap $g = |V(G)|^{1-2\varepsilon}$. By choosing $k = \lceil 1/\varepsilon \rceil$, we obtain a gap of $g' \geq |V(G)|^{(1-2\varepsilon)k}/|V(G)|^4 \geq |V(G)|^{(1-6\varepsilon)k}$. Since $N = |V(H)| = |V(G)|^k$, this gives us the hardness factor $N^{1-6\varepsilon}$, thus proving the first part of the theorem. This reduction runs in time $|V(G)|^{O(1/\varepsilon)} = |\varphi|^{O(1)}$ for constant $\varepsilon > 0$ (since Feige-Kilian reduction runs in polynomial time), thus implying that the hardness result here holds under assumption $\mathsf{P} \neq \mathsf{NP}$.

Similarly, if we start from Thm. 4, we have $g = \frac{n}{2^{\log^{3/4+\varepsilon} n}}$ where $n = |V(G)|$. We plug in the value of $g$ into $g' = g^k/n^3(\log n)^k$. By choosing $k = \log n$, we have $g' \geq g^k/n^{\Theta(\log\log n)} \geq \left(\frac{n}{2^{\log^{3/4+2\varepsilon} n}}\right)^k = \frac{N}{2^{k\log^{3/4+2\varepsilon} n}}$. Since $k = \log n$, we have $\log N = O(k \log n) = O(\log^2 n)$. We obtain the hardness factor $g' \geq \frac{N}{2^{\log^{7/8+O(\varepsilon)} N}}$. The reduction here runs in time $|V(G)|^{O(k)} = |V(G)|^{O(\log|V(G)|)}$. Khot-Ponnuswami reduction has $|V(G)| = 2^{\text{poly}\log|\varphi|}$, and it is randomized with possibly two-sided error. This implies that the running time of the reduction overall is $2^{\text{poly}\log|\varphi|}$. Therefore, this hardness result holds under the assumption that NP does not admit randomized quasi-polynomial time algorithm.

The statements w.r.t. the number of edges follow since $|E(H)| \leq N^2$. $\square$

The rest of this section is devoted to proving Thm. 5. We use a *bipartite double cover* transformation, which transforms any graph $G$ into a bipartite graph $B[G]$ as follows. The nodes of $B[G]$ are $V(B[G]) = \bigcup_{v\in V(G)} \{(v,1),(v,2)\}$, i.e. we make two copies of each vertex $v \in V(G)$. The edges of $B[G]$ are $E(B[G]) = \{(u,1)(v,2) : uv \in E(G)\} \cup \{(u,1)(u,2) : u \in V(G)\}$. Our algorithm simply outputs $H = B[G^k]$. Notice that $|V(H)| = 2|V(G)|^k$.

First let us show the lower bound, which is relatively straightforward to see.

**Lemma 3.** *For any graph $G$, $cp(G) \leq bc(B[G])$*

*Proof.* Let $S_1,\ldots,S_\ell \subseteq V(B[G])$ be the biclique subgraphs that cover $B[G]$. It is sufficient to show how to use these bicliques to define the partition of $G$ into $\ell$ cliques. We name the biclique $H_j = G[S_j]$. For each $j$, we define the vertex set $V_j \subseteq V(G)$ by $V_j = \{v : (v,1)(v,2) \in E(H_j)\}$. First we argue that $G[V_j]$ is a clique in $G$: Consider $u,v \in V_j$ for some $u \neq v$. Since $(u,1)(u,2),(v,1)(v,2) \in E(H_j)$, it must be the case that $(u,1)(v,2) \in E(H_j)$, implying that $uv \in E(G)$. Moreover, the collection of cliques $V_1,\ldots,V_\ell$ together cover graph $G$: For each vertex $v \in V(G)$, an edge $(v,1)(v,2)$ must appear in some $H_{j'}$ (due to the fact that $S_1,\ldots,S_\ell$ are biclique cover). This means that $v \in V_{j'}$. From a collection of cliques $V_1,\ldots,V_\ell$, one can easily modify them into disjoint sets $V'_1,\ldots,V'_\ell$. $\square$

It is easy to see that this inequality implies the lower bound: consider $H = B[G^k]$, so we have $bc(H) \geq cp(G^k) \geq (cp_f(G))^k \geq \left(\frac{cp(G)}{\log|V(G)|}\right)^k$.

Now we need to prove the upper bound that $bp(H) \leq cp(G)^k|V(G)|^3$. We present here a "light" version of our proof, showing a weaker statement that $bc(H) \leq cp(G)^k|V(G)|^3$. This proof captures most of the key ideas we need. The proof of the stronger statement will be contained in the full version.

**Lemma 4.** *For any graphs $G$ and $G'$, $bc(B[G \cdot G']) \leq 2|E(G)| + cp(G)bc(B[G'])$*

Now we can apply Lem. 4 iteratively to get the following, which completes the proof of Thm. 5.

**Lemma 5.** *For any graph $G$ and integer $k$, $bc(B[G^k]) \leq k|V(G)|^2 cp(G^k)$.*

*Proof.* We will argue by induction on $r$ that $bc(B[G^r]) \leq r|V(G)|^2 cp(G^r)$. Notice that this is true for the base case when $r = 1$, i.e. $bc(B[G]) \leq |V(G)|^2 cp(G)$, because the biclique cover number of any graph is at most the number of edges in it. Now assume that the hypothesis holds for all integers up to $r$. By unfolding the term $G^{r+1}$ as $G \cdot G^r$, we can write $bc(B[G^{r+1}])$ as $bc(B[G^{r+1}]) \leq 2|E(G)| + cp(G)bc(B[G^r])$. Applying the induction hypothesis to the second term, we get

$$\begin{aligned} bc(B[G^{r+1}]) &\leq 2|E(G)| + cp(G)r|V(G)|^2 cp(G)^r \\ &\leq |V(G)|^2 + r \cdot cp(G)^{r+1}|V(G)|^2 \\ &\leq (r+1)cp(G)^{r+1}|V(G)|^2 \end{aligned}$$

This implies the proof of the statement. $\square$

### 3.3 Proof of Lemma 4

Recall the statement of the lemma, that $bc(B[G \cdot G']) \leq 2|E(G)| + cp(G)bc(B[G'])$. Let $S_1, \ldots, S_h \subseteq V(B[G'])$ be the biclique cover of $B[G']$. For each $S_j$, we use $G'_j$ to denote the induced subgraph of $S_j$ in $B[G']$ (so $G'_j$ is a clique). We will use these graphs to "suggest" the cover for $B[G \cdot G']$. First, we look at the edges $E(B[G \cdot G'])$ as the union of two edge sets $E_1 \cup E_2$ where

$$E_1 = \{(u, a, 1)(v, b, 2) : u \neq v, uv \in E(G)\}$$

and

$$E_2 = \bigcup_{u \in V(G)} \{(u, a, 1)(u, b, 2) : a = b \vee ab \in E(G')\}.$$

To cover edges in $E_1$, we define the collection of vertices $\{X_{uv}\}_{uv \in E(G)}$ as $X_{uv} = \{(u, a, 1) : a \in V(G')\} \cup \{(v, b, 2) : b \in V(G')\}$. Notice that each $X_{uv}$ is a biclique subgraph of $B[G \cdot G']$: For each pair $(u, a, 1)$ and $(v, b, 2)$ in $X_{uv}$, since $uv \in E(G)$, there must be an edge $(u, a, 1)(v, b, 2)$. Thus, the following claim holds.

*Claim.* The collection $\{X_{uv}\}_{uv \in E(G)}$ covers all edges in $E_1$.

Now we define another collection of bicliques $\{Y_{c,j}\}$ to cover edges in $E_2$ as follows. Let $C_1, \ldots, C_\ell$ be the partition of vertices of $G$ into cliques. For each clique $c = 1, \ldots, \ell$, for each $j = 1, \ldots, h$, define a subset of vertices $Y_{c,j} \subseteq V(B[G \cdot G'])$ where $Y_{c,j} = \{(u, a, 1) : u \in C_c, (a, 1) \in S_j\} \cup \{(u, b, 2) : u \in C_c, (b, 2) \in S_j\}$. Now we verify that the induced subgraph of each $Y_{c,j}$ is biclique: For any pair of vertices $(u, a, 1), (v, b, 2) \in Y_{c,j}$,

- If $u = v$, then it must hold that $(a, 1)(b, 2) \in E(G'_j)$ (because both $(a, 1)$ and $(b, 2)$ belong to biclique $S_j$). There are two cases again. If $a = b$, we have an edge $(u, a, 1)(u, a, 2) \in B[G \cdot G']$ by definition; otherwise, if $a \neq b$, there must be an edge $ab \in E(G')$, implying that $(u, a, 1)(u, b, 2)$ is an edge in $B[G \cdot G']$.
- If $u \neq v$, the fact that both $u$ and $v$ belong to the same clique $C_c$ means that an edge $uv \in E(G)$, implying that $(u, a, 1)(v, b, 2)$ is an edge in $B[G \cdot G']$.

*Claim.* The collection of bicliques $Y_{c,j}$ covers all edges in $E_2$.

*Proof.* Fix some $u \in V(G)$. Consider an edge $(u, a, 1)(u, b, 2) \in E_2$. Let $C_c$ be the clique that contains vertex $u$. Since $ab \in E(G')$ or $a = b$, we have $(a, 1)(b, 2)$ as an edge in $B[G']$. Therefore, it is covered by some biclique $G'_j$, i.e. $(a, 1), (b, 2) \in S_j$. This implies that both $(u, a, 1)$ and $(u, b, 2)$ belong to $Y_{c,j}$, hence covered. $\square$

## 4 Algorithmic Results

We will now give two approximation algorithms for BicliqueCover. Thereby, we achieve two mutually non-dominating approximation guarantees in terms of the number of nodes and edges, respectively.

### 4.1 An Approximation Guarantee of $O(n/\sqrt{\log(n)})$

We first describe a simple approximation algorithm for BicliqueCover that achieves a performance ratio of $O(n_U/\sqrt{\log(n_U)})$ where $n_U$ is the number of left vertices in the bipartite input graph $G = (U \cup V, E)$ (we assume w.l.o.g that the left side of the graph is the smaller one, i.e. $|U| \leq |V|$). Moreover, we will apply exactly the same scheme to solve BicliquePartition, thereby achieving the same performance guarantee for BicliquePartition.

The main idea behind the algorithm is to split the left vertex set $U$ in parts of equal size $r$ (to be fixed later) and run an $\alpha(r)$-approximation algorithm for finding a biclique cover in each of these subgraphs. The results of all $n_U/r$ subproblems are then put together to form a biclique cover of the whole graph $G$. This also works for biclique partition, as the subgraphs are edge-disjoint. The following theorem relates the approximation guarantee for the subproblems to the guarantee for the overall problem.

**Lemma 6.** *Let $G = (U, V, E)$ be a bipartite graph with $n_U = |U| \leq |V|$. If we can solve* BicliqueCover *on a graph $G'$ with $r$ left vertices with an approximation guarantee of $\alpha(r)$, then we can solve the problem on $G$ with approximation guarantee $\frac{n_U}{r}\alpha(r)$. The same holds for* BicliquePartition.

*Proof.* Partition $U$ arbitrarily into $n_U/r$ sets $U_1, \dots, U_{n_U/r}$ of size $r$ and run the approximation algorithm with performance guarantee $\alpha(r)$ on the subgraphs induced by the sets $U_i$ and their neighborhoods. Let $G_i$ denote the $i$-th subgraph and $APX_i$ the size of the solution produced by the approximation algorithm on $G_i$. Furthermore, let $\mathsf{OPT}_i$ be the size of the optimal solution on subgraph $G_i$ and $\mathsf{OPT}$ be the size of the optimal solution for $G$. Notice that the union of the biclique covers of the subgraphs gives a biclique cover for $G$. Therefore, we have that the size of this combined solution is $APX = \sum_{i=1}^{n_U/r} APX_i \leq \alpha(r) \sum_{i=1}^{n_U/r} \mathsf{OPT}_i \leq \alpha(r) \frac{n_U}{r} \mathsf{OPT}$. The last inequality follows as the optimal solution of a subgraph of $G$ is at most as large as the optimal solution of $G$. This analysis also applies to BicliquePartition. □

**Theorem 6.** *There are $O(n/\sqrt{\log n})$ approximation algorithms for* BicliqueCover *and* BicliquePartition.

*Proof.* For solving the subproblems on $G' = (U', V', E')$ with $r = |U'|$ left vertices, we run a brute-force algorithm: Enumerate all $2^r$ subsets of the left vertices and enumerate all $r$-tuples of such subsets. Such a subset $S \subseteq U'$ induces a biclique together with the intersection of the neighborhoods of all vertices

$v \in S$. Then return the smallest tuple of vertex sets that covers all edges. For BicliquePartition, additionally ensure that the bicliques are edge-disjoint. As the optimal solution needs at most $r$ bicliques (simply take all the bicliques induced by one of the left vertices) and we enumerate all bicliques of the graph by enumerating all subsets of left vertices, this will return the optimal solution. Hence the approximation factor on the subproblems is $\alpha(r) = 1$. Thus, for the whole algorithm on $G$, we get a guarantee of $\frac{n_U}{r} \leq \frac{n}{r}$. The running time of the brute-force algorithm is $O((2^r)^r)$, hence by choosing $r = \sqrt{\log(n)}$ we get a polynomial running time of the algorithm and a guarantee of $O(n/\sqrt{\log(n)})$. □

### 4.2 An Approximation Guarantee w.r.t. the Number of Edges

A different approach to obtain an approximation guarantee, which dominates the previous one on sparse graphs, is obtained via the following construction.

**Definition 1.** *For a given undirected graph $G = (V, E)$, the* conflict graph *$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains a node for each edge of $G$, i.e. $|\mathcal{V}| = |E|$. Two nodes of $\mathcal{G}$ are connected by an edge if and only if the two corresponding edges of $G$ are not contained in a common biclique.*

A node coloring of $\mathcal{G}$ corresponds to an edge coloring of $G$ such that each color-class is contained in a common biclique. Thus, the chromatic number of $\mathcal{G}$ is equal to $bc(G)$ and we can use [16] to obtain a guarantee in $O\left(\frac{m \log^2 \log m}{\log^3 m}\right)$. Together with Thm. 6, this concludes the proof of Thm. 2.

However, we present another perspective, which not only gives algorithmic insights but also leads to an improved hardness result for MaxWeightedBiclique. To this end, recall that the chromatic number of $\mathcal{G}$ is equal to $cp(\overline{\mathcal{G}})$, so that a greedy algorithm that covers $\overline{\mathcal{G}}$ with cliques also covers $G$ with bicliques.

Thus, we analyze the family of greedy algorithms that pick a biclique in each iteration containing as many uncovered edges as possible until every edge is covered. To this end, we reexamine the relation between master and slave problems in Johnson's framework [19] under the premise that the approximation guarantee $\alpha(\cdot)$ is an increasing function on the number of uncovered elements. That is, the approximation guarantee improves over iterations as the number of uncovered edges shrinks. Hence, our master problem is BicliqueCover and its slave problem is the problem of finding the heaviest biclique with edge-weights in $\{0, 1\}$ being 0 if an edge is already covered and 1 if not. Our result is summarized in the following theorem, whose proof will appear in the full version of this paper.

**Theorem 7.** *Let $G = (V, E)$ be a bipartite graph. If there is an $\alpha$-approximation algorithm for* MaxWeightedBiclique*, then there is a greedy algorithm that computes a* BicliqueCover *of size*

$$O\left(\alpha \log\left(\frac{|E|}{\alpha}\right) bc_f(G)\right),$$

*where $bc_f(G)$ is the fractional biclique cover number.*

Such an $\alpha$-approximation can be obtained from an approximation algorithm for MaxClique operating on the complement of a conflict graph. By dropping all the nodes of $\overline{\mathcal{G}}$ that correspond to edges with weight 0 and finding an approximation of the largest clique in the remainder, we obtain a set of edges of $G$ that belongs to a common biclique, which has a weight of at least the maximum weight divided by $\alpha$. Using the MaxClique algorithm of Feige [11], we obtain an approximation factor for $\{0,1\}$-weighted biclique in $O\left(\frac{m \log^2 \log m}{\log^3 m}\right)$. This is also essentially the best one can hope for as our new hardness result shows.

**Corollary 2.** *There is no poly-time algorithm to approximate* MaxWeightedBiclique *within factors of* $n^{1-\varepsilon}$ *and* $m^{1/2-\varepsilon}$*, respectively, for all* $\varepsilon > 0$ *unless* $P = NP$*, or within a factor of* $O\left(\frac{\min\{n,\sqrt{m}\}}{2^{\log^{7/8+\varepsilon} n}}\right)$ *for any* $\varepsilon > 0$ *unless* $\mathsf{NP} \subseteq \mathsf{BPTIME}(2^{poly \log n})$*. This holds even when edge-weights are in* $\{0,1\}$*.*

A further consequence of Thm. 7 is that $bc(G) = O(\log(n) bc_f(G))$, which yields the following corollary.

**Corollary 4.** *It is NP-hard to approximate the fractional biclique number within* $n^{1-\varepsilon}$ *or* $m^{1/2-\epsilon}$ *for all* $\varepsilon > 0$*.*

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