# A Combinatorial Algorithm for the 1-Median Problem in $\mathbb{R}^d$ with the Chebyshev-Norm


Johannes Hatzl[a,*], Andreas Karrenbauer[b]

[a] *Department of Optimization and Discrete Mathematics, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria.*

[b] *Department of Computer and Information Science, Zukunftskolleg, University of Konstanz, Box 216, 78457 Konstanz, Germany.*



## Abstract

We consider the 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm: Given $n$ points with non-negative weights, find a point that minimizes the sum of the weighted distances to the given points. We propose a combinatorial algorithm for this problem by reformulating it as a fractional $b$-matching problem. This graph-theoretical problem can be solved by a min-cost-flow algorithm. Moreover, we show that there is a 1-median, which is half-integral, provided that the points have integral coordinates.



*Keywords:* facility location, 1-median problem, fractional $b$-matching


## 1. Introduction

Location problems are a well studied branch of operations research and are also important from a practical point of view in order to minimize transportation cost or to serve clients in a best possible way. The roots of location problems can be seen in an essay of Pierre de Fermat where he posed the question: Where should a single facility be placed in the plane such that the sum of the distances from a given set of three points is minimized. In the last century, the economist Weber [9] extended the model by allowing an arbitrary number of points and by assigning weights to the points. In his model the task is to find a point that minimizes the sum of the weighted distances. In modern terminology this problem is the 1-median problem in the Euclidean plane and is nowadays also known as the Fermat-Weber problem. It has been studied intensively by many researchers (see e.g. [3] and the references therein).

In the last decades, a lot of different models have been considered. The number of facilities that are required to be located was not fixed to one any more and the space in which the facilities can be placed was varied. Continuous location problems (e.g. problems in $\mathbb{R}^d$) and network location problems, where the facilities can be opened on a graph, are the most studied problems. Moreover, different objective functions can be investigated. One can analyze the sum of the weighted distances (median problems) or one may be interested in minimizing the largest weighted distance to a vertex (center problems). Recently, Nickel and Puerto [6] introduced the ordered median function, which can be seen as a generalization of the median and center problems.

This paper focuses on the weighted 1-median problem in $\mathbb{R}^d$ where the distance between two points is measured by the Chebyshev-norm. So far this problem is only well understood for $d = 2$. In this special case, the Chebyshev-metric and the Manhattan-metric are very closely related and using this fact a linear time algorithm is given in [4]. However, to the best of our knowledge for $d \geq 3$ no combinatorial algorithm is known so far. In [8], an algorithm is discussed for the three-dimensional case that finds only near-optimal solutions in reasonable computational time.


*Corresponding author.

Emailadresses: hatzl@opt.math.tugraz.at; andreas.karrenbauer@uni-konstanz.de

In this paper, we show that for $d \in \mathbb{N}$ the location problem can be reformulated as a fractional $b$-matching problem and can thus be solved by a combinatorial algorithm.

This paper is organized a follows: In the next section the problem under consideration is rigorously defined. Moreover, it is written as a linear programming problem which implies that the problem can be solved in polynomial time. Afterwards it is shown that using Fourier-Motzkin-elimination the problem is equivalent to the fractional $b$-matching problem on the complete graph. This graph-theoretical problem can be solved as a min-cost-flow problem in a bipartite graph (see e.g. [1]). In order to make the paper self-contained we shortly review the min-cost-flow instance in the last section.

## 2. Problem Formulation

In this section, we define the 1-median Problem in $\mathbb{R}^d$ with the Chebyshev-norm formally: Given $n$ distinct points $P_1, \ldots, P_n$ with $P_i = (x_1^i, \ldots, x_d^i) \in \mathbb{R}^d$ for $i = 1, \ldots, n$ and associated non-negative weights $w_i \geq 0$ the task is to find a point $P^* = (x_1^*, \ldots, x_d^*) \in \mathbb{R}^d$ such that

$$\sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty \geq \sum_{i=1}^{n} w_i \, \|P_i - P^*\|_\infty$$

holds for all $P \in \mathbb{R}^d$, where $\|P_i - P^*\|_\infty := \max\left(|x_1^i - x_1^*|, \ldots, |x_d^i - x_d^*|\right)$ is the Chebyshev-norm.

For $d = 2$ there is a well known optimality criterion. Given a point $P_0 = (x_1^0, x_2^0)$ we define the sets

$$X^1_\sim := \{i : x_1^i - x_1^0 \sim x_2^i - x_2^0\} \qquad \text{and} \qquad X^2_\sim := \{i : x_1^i - x_1^0 \sim -(x_2^i - x_2^0)\}$$

and

$$w(X^1_\sim) := \sum_{i \in X^1_\sim} w_i \qquad \text{and} \qquad w(X^2_\sim) := \sum_{i \in X^2_\sim} w_i$$

where $\sim$ is any of the relations $<, \leq, >, \geq$. Using this notation we can state the following theorem.

**Theorem 2.1** (see e.g. [2])**.** *The point $P_0 = (x_1^0, x_2^0)$ is a 1-median if and only if the following four inequalities are satisfied*

$$w(X^1_<) \leq w(X^1_\geq) \qquad \textit{and} \qquad w(X^2_<) \leq w(X^2_\geq)$$

*and*

$$w(X^1_>) \leq w(X^1_\leq) \qquad \textit{and} \qquad w(X^2_>) \leq w(X^2_\leq).$$

Theorem 2.1 can be used to derive a linear time algorithm for the case $d = 2$ (see e.g. [4]). However, for $d \geq 3$ the optimality criterion is much more complex as shown in [5]) and does not lead to an algorithm.

In order to develop a combinatorial algorithm for higher dimensions we rewrite the location problem in the $d$-dimensional space as a linear programming problem. We can reformulate

$$\min_{P=(y_1,\ldots,y_d)\in\mathbb{R}^d} \sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty$$

as

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i \, z_i \\
\text{s.t.} \quad & z_i = \max\left(|x_1^i - y_1|, \ldots, |x_d^i - y_d|\right) && i = 1, \ldots, n \\
& y_k \in \mathbb{R}, \;\; z_i \in \mathbb{R} && i = 1, \ldots, n, \;\; k = 1, \ldots, d.
\end{aligned}$$

It is a well known fact that minimizing the sum of absolute values with linear constraints can be transformed to an LP in standard form which has the following form:

$$\min \quad \sum_{i=1}^{n} w_i\, z_i \tag{1}$$

$$\text{s.t.} \quad z_i + y_k \geq x_k^i \qquad i = 1, \dots, n, \;\; k = 1, \dots, d \tag{2}$$

$$z_i - y_k \geq -x_k^i \qquad i = 1, \dots, n, \;\; k = 1, \dots, d \tag{3}$$

$$y_k \in \mathbb{R}, \;\; z_i \in \mathbb{R} \quad i = 1, \dots, n, \;\; k = 1, \dots, d. \tag{4}$$

Note that the problem can be formulated as linear programming problem and can thus be solved in polynomial time. In the next section, we show that the LP given in (1)—(4) is equivalent to the fractional $b$-matching problem and can even be solved by a combinatorial algorithm.

## 3. Reformulation as fractional $b$-matching problem

Due to the fact that the variables $y_k$ do not appear in the objective function of the problem given in (1)—(4), we use the well known Fourier-Motzkin elimination technique to get rid of these variables in the constraints. More precisely, if $(y, z)$ is a feasible solution of the problem we know that

$$x_k^i - z_i \leq y_k \leq x_k^i + z_i$$

holds for all $k = 1, \dots, d$ and $i = 1, \dots, n$. In particular,

$$\max_{i=1,\dots,n} (x_k^i - z_i) \leq y_k \leq \min_{i=1,\dots,n} (x_k^i + z_i) \tag{5}$$

is true for all $k = 1, \dots, d$. Moreover, by adding the inequalities (2) and (3) it can be seen that $z$ also satisfies

$$z_i + z_j \geq x_k^i - x_k^j$$

for all $i, j = 1, \dots, n$ and $k = 1, \dots, d$, which is equivalent to

$$z_i + z_j \geq d_k^{ij},$$

if we define $d_k^{ij} := |x_k^i - x_k^j|$. Note that $d_k^{ij} = d_k^{ji}$ follows directly from the definition. Moreover, $d_k^{ij}$ is non-negative and $d_k^{ij} = 0$ if and only if $i = j$. Thus, if $(y, z)$ is feasible for the constraints (2)—(4) then $z$ is also feasible for following system of inequalities:

$$z_i + z_j \geq d^{ij} \qquad \forall j = 1, \dots, n, \;\; i = j+1, \dots, n \tag{6}$$

$$z_i \geq 0 \qquad \forall i = 1, \dots, n \tag{7}$$

where $d^{ij} := \max_k d_k^{ij}$.

On the other hand, if there is a vector $z$ satisfying (6) and (7) we can easily get a feasible solution $(y, z)$ of the linear programming problem (1)—(4) by setting $y_k$ to any value between $\min_i(z_i + x_k^i)$ and $\max_i(x_k^i - z_i)$. It follows from the discussion above that this is always possible. Hence, problem (1)—(4) is equivalent to the following linear programming problem

$$\min \quad \sum_{i=1}^{n} w_i\, z_i \tag{8}$$

$$\text{s.t.} \quad z_i + z_j \geq d^{ij} \quad j = 1, \dots, n, \;\; i = j+1, \dots, n \tag{9}$$

$$z_i \geq 0 \qquad i = 1, \dots, n \tag{10}$$

Before we consider the dual of this problem let us recall the definition of the $b$-matching problem on a complete graph. Let $G = (V, E)$ be a complete graph with a function $b : V \to \mathbb{N}$. A $b$-matching is an assignment of non-negative integer values $\alpha_{ij}$ for all $(i, j) \in E$ such that for each vertex $i$ the sum of the values on the edges incident with $i$ is not more than $b_i$. The task of the $b$-matching problem adapted to our setting is to find a longest $b$-matching, where the length of a $b$-matching $\alpha_{ij}$ is given by

$$\sum_{(i,j)\in E} d^{ij}\alpha_{ij}.$$

The integer programming formulation of this graph-theoretical problem is given by

$$\max \quad \sum_{(i,j)\in E} d^{ij}\alpha_{ij} \tag{11}$$

$$\text{s.t.} \quad \sum_{(i,j)\in E} \alpha_{ij} \le b_i \quad \forall\, i \in V \tag{12}$$

$$\alpha_{ij} \in \mathbb{N} \qquad \forall\, (i,j) \in E \tag{13}$$

It is easy to see that if we introduce for each constraint in (9) a non-negative dual variable $\alpha_{ij}$ the dual linear programming problem of the location problem given in (8)—(9) is exactly the linear relaxation of the problem given in (11)—(13) (with $b_i = w_i$). Thus, in order to find a 1-median it suffices to solve the fractional $b$-matching problem, i.e., the problem (11)—(12) with (13) relaxed to $\alpha_{ij} \ge 0$ for all $(i,j) \in E$.

## 4. A Min-cost-flow algorithm for the fractional *b*-matching problem

In this section, we show how the fractional $b$-matching problem can be solved by a min-cost flow algorithm in a bipartite network. This algorithm has already been suggested by Antsee[1], who gave an algorithm to compute a solution for the fractional $b$-matching problem, which was then used to find an optimal solution for the $b$-matching problem.

Let us define the bipartite graph $G = (V_1 \cup V_2, E)$ where $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{v'_1, \ldots, v'_n\}$ and there is an edge $(v_i, v'_j)$ for all $i, j = 1, \ldots, n$. Moreover, each edge $(v_i, v'_j)$ has infinite capacity and the cost are given by $-d^{ij}$. Note that the edges $(v_i, v'_i)$ have 0 cost. They serve as slack variables to transform (12) into equality constraints. That is, for each vertex $v_i \in V_1$ we define a supply of $b_i$ and for each vertex $v_j \in V_2$ we define a demand of $b_j$. A min-cost-flow instance of that kind is sometimes also called a Hitchcock-Transportation-Problem and can be solved in $O(n^3 \log n)$ by a combinatorial algorithm of Orlin [7]. It is well known that, for integral supplies and demands, this algorithm always computes an integral optimal flow $f^* : E \to \mathbb{N}$ to this problem. However, more important for us is the fact that even for an integral cost function, there exists a feasible integral potential, i.e., two functions $\pi : V_1 \to \mathbb{N}$ and $\pi' : V_2 \to \mathbb{N}$ such that $\pi(i) + \pi'(j) \le d^{ij}$ holds for all edges $(i,j)$. Using these values Antsee [1] has shown that

$$z_i^* := \frac{1}{2}\left(\pi(i) + \pi'(i)\right) \tag{14}$$

and

$$\alpha_{ij}^* := \frac{1}{2}\left(f^*(v'_i, v_j) + f^*(v_i, v'_j)\right)$$

are optimal solutions to (8)—(10) and the linear relaxation of (11)—(13). Using these results we can finally state the following theorem.

**Theorem 4.1.** *If all given points of the 1-median problem have integral coordinates, then there exists a half-integral 1-median $P^*$.*

*Proof.* If the coordinates and the weights of the given points are integral, it follows from (14) that there exists an optimal solution $(y, z)$ of (8)—(10) in which $z$ is half-integral. Due to the fact that we can choose $y$ such that (5) is satisfied it follows that we can also construct a half-integral vector $y$ which gives the coordinates of a 1-median. The same holds for rational weights of the points (i.e. the values $w_i$ and $b_i$, respectively) as we may scale them by a common denominator without changing the 1-median. Moreover, even for irrational weights the statement follows from LP theory. Observe to this end that the polyhedron defined by (9) and (10) is rational. Therefore all extreme points are rational and moreover they have rational supporting hyperplanes. Hence, we obtain integral weights for the points that yield the same 1-median as the original weights. Thus, the claim follows from the consideration above. □


### Acknowledgement

The first author is partially supported by the Austrian Science Fund Project P18918-N18 *Efficient solvable variants of location problems.*


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