Multiple pass streaming algorithms for learning mixtures of distributions in $\mathbb{R}^d$

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October 2, 2007

Theoretical Abstractions for Massive Data Set Computation

- The input on disk/storage is modeled as a read-only array. Elements may only be accessed through a sequential pass.
  - Input elements can be arbitrarily ordered.
- Main memory is modeled as extra space used for intermediate calculations.
- Algorithm is allowed some extra computing time before and after each pass.
- Goal: Design algorithms that minimize memory usage, number of passes, extra computing time.

Models of Computation

- Streaming Model: Algorithm may make a single pass over the data. Space must be $o(n)$.
- Pass-Efficient Model: Algorithm may make a small, constant number of passes. Ideally, space is $O(1)$.
- Other models: external memory, sublinear algorithms

Note that Pass-Efficient is more flexible than streaming, but not suitable for "streaming" data arriving that is processed immediately and then "forgotten".

Why do we need multiple passes?

- Is it better, in terms of other resources, to make 3 passes instead of 1 pass?
- Example: Find the mean of an array of n integers.
  - 1 pass requires $O(1)$ space.
  - More passes don't help.
- Example: Find the median. [MP '80]
  - 1 pass algorithm requires $\Omega(n)$ space.
  - 2 pass needs only $O(n^{1/2})$ space.
- We study the trade-off between passes and memory.

Generative clustering model: Mixtures of k distributions in $\mathbb{R}^d$

- k probability distributions $F_1, \ldots, F_k$, each with weight $w_i > 0$ such that $\sum w_i = 1$.
- To generate a sample x from the mixture: Pick a distribution F_i with probability w_i. Draw a point according to F_i.
- If each $F_i : \Omega \to \mathbb{R}$ is a density function, then density of mixture is $F(x) = \sum_i w_i F_i(x)$.

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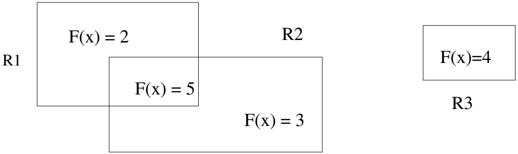
- If each $F_i : \Omega \to \mathbb{R}$ is a density function, then density of mixture is $F(x) = \sum_i w_i F_i(x)$.
- Place the samples in a data stream, and order arbitrarily.
- Goal of algorithm: from data stream, approximate F with a function G. Error measured by L^1 norm: $\int_\Omega |F - G|$.

Mixtures of k uniform distributions in $\mathbb{R}^d$

- Each mixture component F_i is a uniform distribution over some cell R_i in $[0,1]^d$: $R_i = (a_1, b_1) \times \ldots \times (a_d, b_d) = \{x \in \mathbb{R}^d | a_i < x_i < b_i, i = 1, \ldots, d\}$.



Main Result: mixture of k uniform distributions in $\mathbb{R}^d$

- For any $\ell > 0$, a $2\ell + 1$ pass algorithm that, with probability at least $1 - \delta$, learns F to within error ϵ such that
  - Memory required:

$$\tilde{O}\left(\frac{k^3 d^3}{\epsilon^{2/\ell}} \log(1/\delta) + (2k)^d\right)$$

  - Sample complexity: Requires $\tilde{\Omega}((wkd/\epsilon)^{O(d)})$ samples in data stream.
- This exhibits a trade-off between the number of passes and memory.
  - Error stays the same, but the memory decreases significantly with each extra pair of passes!
  - If 3 passes, requires $\approx 1/\epsilon^2$ space, 5 passes: $\approx 1/\sqrt{\epsilon}$, 9 passes $\approx 1/\sqrt{\epsilon}$.
  - Suffers from curse of dimensionality.

Main Result (second interpretation)

- For any $\ell > 0$, a $2\ell + 1$ pass algorithm that, with probability $1 - \delta$, learns F to within error ϵ^ℓ such that
  - Memory required:

$$\tilde{O}\left(\frac{k^3 d^3}{\epsilon^2}\log(1/\delta) + \frac{\ell k d^2}{\epsilon}\log(1/\delta) + (2k)^d\right)$$

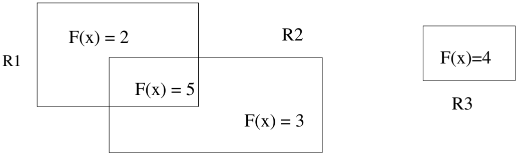
- A few extra passes doesn't affect the memory requirement much, but the error drops exponentially!
- Rest of talk will focus on this error guarantee.

Related Work

- Same problem, $d = 1, 2$ dimension [Chang& Kannan '06], [Guha & McGregor '07].
- Learning mixtures of Gaussian distributions in $\mathbb{R}^d$. Not streaming. [Dasgupta '99, Arora & Kannan '05, Vempala & Wang '04, KSV '06]
- Multidimensional histograms in data streams. [TGIK '02]
  - Streaming alg, similar to our problem, but instead of assuming generative model of data, just find the best fit function.
  - also suffers from curse of dimensionality.

Preliminaries: Structure of cells in $\mathbb{R}^d$

- Cell $R_i = \{x \in \mathbb{R}^d | a_i \leq x_i \leq b_i\}$ has $2d$ boundary faces (for $d = 2$, edges of the rectangle).
- For $m \in [d]$, the two m-boundary faces are: $\{x \in R_i | x_m = a_m\}$ and $\{x \in R_i | x_m = b_m\}$.
- Each boundary face can be thought of as a $d-1$ dimensional cell that is completely contained in a $d-1$ dimensional hyperplane with fixed mth coordinate.

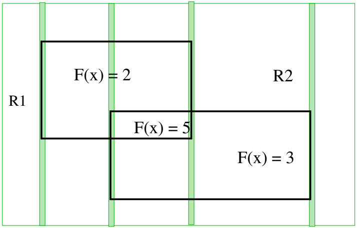


(x, y), $\rightarrow$ first coordinate, $\uparrow$ second coordinate.

Subroutine 1: Find those hyperplanes!

Input: m, data stream.
Find the set of hyperplanes with fixed mth component that contain all m-boundary faces.

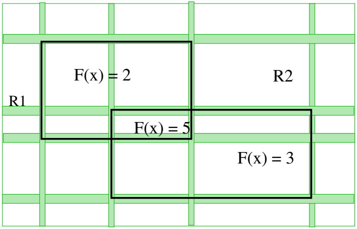
- Uses 2ℓ passes, space $\tilde{O}(k^3 d^2/\epsilon^2 + kd\ell/\epsilon)$.
- Get $2k$ hyperplanes that contain all m-boundary faces.
- Warning: real subroutine does this with some error $\approx \epsilon^\ell/d$.



More details on how to accomplish this later.

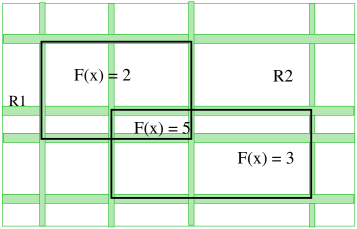
Using SR1 to learn the mixture

- Run SR1 for all $m = 1, \ldots, d$, in parallel. 2ℓ passes.



Using SR1 to learn the mixture

- Run SR1 for all $m = 1, \ldots, d$, in parallel. 2ℓ passes.
- The output hyperplanes partition bounding box into a set of cells $\mathcal{C}$ such that F is constant when restricted to each cell. Number of such cells: $|\mathcal{C}| \leq (2k)^d$.



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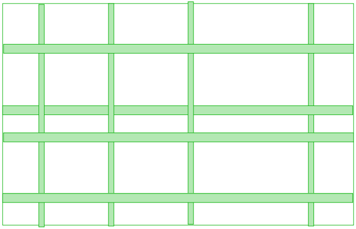
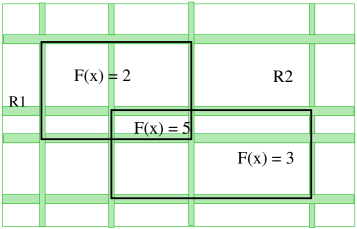


Using SR1 to learn the mixture, continued



In one more pass: count the number of points of data stream X that falls in each $C \in \mathcal{C}$.

- Since F is constant on each C, $|X \cap C|/(|X|\mathrm{Vol}(C))$ will approximate $F(x)$, $x \in C$.
- With $|X|$ sufficiently large, approximation of F will have L^1 error at most ϵ^ℓ.

Memory used: $\tilde{O}(k^3d^3/\epsilon^2 + kd^2\ell/\epsilon + (2k)^d)$.
Passes used: $2\ell + 1$.

Finding boundary hyperplanes: Subroutine 1

Iterative algorithm, in ℓ pairs of passes.

1. First pass: Take a sample of size $s = O(d^2 k^2/\epsilon^2)$ from the data stream.
   - Partition the bounding box R into a set of cells $\{P_i\}$ such that each m-boundary face is completely contained in one P_i.
   - Using the sample, can do this such that $\int_{P_i} F = \Theta(\epsilon/dk)$, whp.
   - Number of cells will be $\Theta(dk/\epsilon)$.

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2. Second pass: Test each partition cell P_i to see if P_i contains an m-boundary face.
   - Test algorithm requires $\tilde{O}(d\ell)$ space to make.

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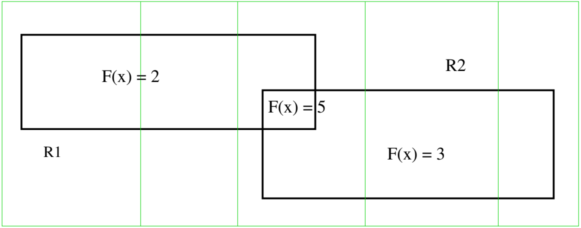
Warning: real algorithm more complicated.

An example for $d = 2, m = 1$.



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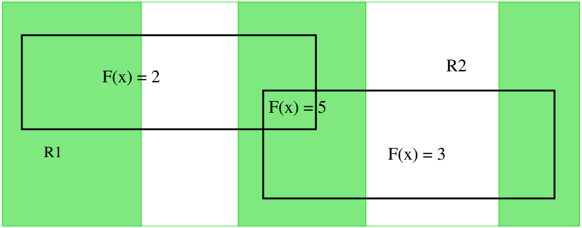

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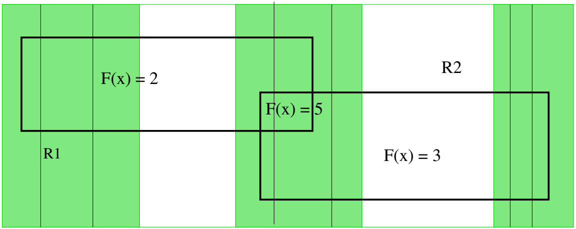

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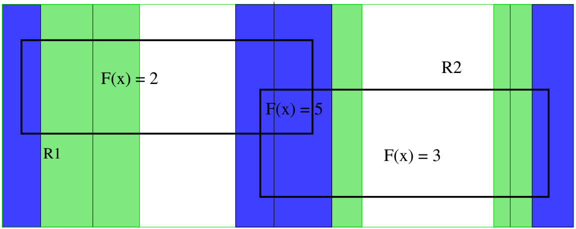

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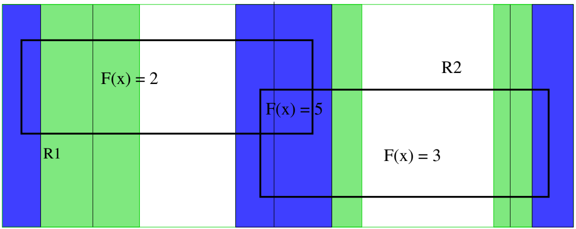

What happens when you iterate?

- Volume of each partition cell will decrease by at least a multiplicative factor of ϵ.
- After ℓ iterations (2ℓ passes), each partition cell that contains a boundary hyperplane has volume at most $\approx \epsilon^{\ell}/(dk)$.
- Thus, after ℓ iterations, will have isolated each boundary face to within a partition cell of volume $\epsilon^{\ell}/(dk)$.

What's the space complexity?

- Each iteration takes a sample of size s and runs the test algorithm for kd/ϵ cells.
- $\tilde{O}(s + d\ell/\epsilon) = \tilde{O}(k^2 d^2/\epsilon^2 + d\ell)$.
- Recursion only occurs on at most $2k$ cells in parallel. Total memory: $\tilde{O}(k^3 d^2/\epsilon^2 + d\ell k)$.

Testing a partition cell for boundaries: The engine

- Fundamental one pass algorithm [Indyk 1999] for computing the ℓ_1 length of a vector $v \in \mathbb{R}^n$ given as a stream of dynamic updates.
- Input: a data stream that consists of pairs $\langle a, i \rangle$, $i \in [n]$ and $a \in (-M, M)$.
- Suppose we have a vector $v \in \mathbb{R}^n$. Each pair is an update of v with semantics: add a to ith component of v.
- Approximate in one pass $||v||_1$ or $||v||_2$ with $o(n)$ space.

Thanks for your time!