The Space Complexity of Pass-Efficient Algorithms for Clustering

Kevin L. Chang[1] Ravi Kannan[1]

[1]Department of Computer Science
Yale University

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Streaming Algorithms

TCS model for abstracting computation on massive data sets.

- Input is placed in a read-only array.
- Elements in the array can only be accessed by a single pass through the entire array.
- Algorithm allowed a sublinear amount of working memory in order to perform intermediate calculations, store sketches, etc.
- Optimize working memory.

First few applications: median finding [MP80], statistics [AMS96]

Pass-Efficient Algorithms

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- Memory usage should be a constant, independent of n.
- **We will be concerned with trading the number of passes and memory.**

Generative Clustering: Mixtures of Distributions

- k distributions $F_1, \ldots, F_k$ over the same universe Ω.
- Weight $w_i \geq 0$ for each F_i, such that $\sum_{i=1}^{k} w_i = 1$.
- We “add” these k distributions to create a new distribution called a *mixture*.
- The mixture is given by: $\sum w_i F_i$.

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Can generalize to mixtures of other types of distributions as well.

Motivation

- Unsupervised learning of parameters of generative mixture models from samples is a popular statistical tool.
- Many theory papers consider mixtures of Gaussian distributions in high dimension: learning means, mixing weights, and covariance matrices [Dasgupta99], [AK01], [VW02], etc

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- A 2ℓ pass algorithm with error ϵ^ℓ that uses $\tilde{O}(k^3/\epsilon^2 + \ell k/\epsilon)$ RAM for learning mixtures of uniform distributions.

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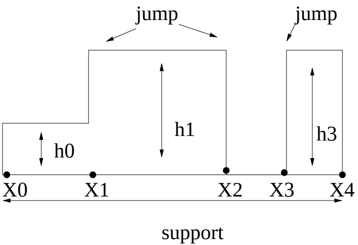
These algorithms have failure probabilities of $1 - \delta$.

What does a mixture of k uniform distributions look like?



Thus, our algorithm will assume the sample is drawn from distribution with density given by a step function with $2k - 1$ jumps.

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- If you want ϵ^ℓ error, then you will need to store $\Omega(\frac{1}{\epsilon^{2\ell}})$ counters.
- Too much! We can do much better by making a few more passes.

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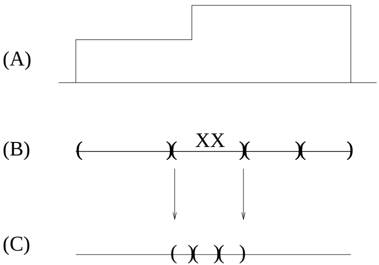
Requires $|X| = \tilde{\Omega}\left(\frac{k^6}{\epsilon^{6\ell}} \cdot \ell\right)$ points from F.

Zooming In



- A Function F.
- B Partition the domain into intervals.
- C Recurse on non-constant interval. It is sampled at a higher rate.

Why it works

- Memory requirement is small: since we only recurse on $2k$ jumps, the number of bins created at each iteration is at most $O(k^2/\epsilon)$.
- Easy to learn F when it is constant. Our estimate is very accurate.
- The weight of bins decreases exponentially at each iteration.
- At the ℓth iteration, bins have weight $\epsilon^\ell/4k$.
- Thus, we can estimate F as 0 on $2k$ bins where there is a jump, and incur a total error of at most $\epsilon^\ell/2$.

Generalizations

Can generalize above algorithm to the following problems, with roughly the same memory usage:

- Uniform distributions over $\mathbb{R}^2$: F_i is uniform over some axis-aligned rectangle: $(a_i, b_i) \times (c_i, d_i) \subset \mathbb{R}^2$.
- Linear distributions over $\mathbb{R}$: The density of F_i is a linear function over some continuous interval $(a_i, b_i) \subset \mathbb{R}$.

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Subproblem: Suppose H is the pdf of a mixture of k uniform distributions over an interval I, with samples in input array X. Determine if $\int_I |H - 1| \leq \epsilon^\ell / 2k$.

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1. Partition I into many bins of equal length. Number of bins is $5k^2/\epsilon^\ell$.
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Problem: This will take at least $\frac{5k^2}{\epsilon^\ell}$ bits of memory. Too much!

Maintaining ℓ_1 length of a vector

[Indyk00] designed a streaming algorithm for maintaining the ℓ_1 length of a vector v.

- Data stream consists of pairs (i, a), where $i \in [n]$, $a \in \{-M, \ldots, M\}$.
- Vector $v \in \mathbb{R}^n$ given by: $v_i = \sum_{(i,a)} a$.
- With probability $1 - \delta$, Indyk's algorithm will approximate $||v||_1$ within a constant factor in one pass using at most $O(\log M \log(n/\delta) \log \delta)$ bits of memory.

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Uses $O\left((\log k + \ell \log(1/\epsilon)) \log(1/\delta)\right)$ bits of memory.

Proving Lower Bounds for the Generalized Learning Problem

Prove lower bounds for slightly stronger problem:

- F is the pdf of a mixture of $1/\epsilon^{\ell}$ uniform distributions.
- Let $t \in [0,1]$ be the largest number such that F is a step distribution with at most k steps on $[0,t]$.
- Find a function G and number $t' > t$ such that $\int_0^{t'} |F - G| < \alpha$.

THEOREM: Any ℓ-pass randomized algorithm that solves the Generalized Learning Problem for $k = 3$ and error ϵ^{ℓ} must use at least $\Omega(1/\epsilon^{1/2}c^{-2\ell+1})$ bits of memory.

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There exists an ℓ-pass algorithm that will solve the problem using at most $\tilde{O}(1/\epsilon^4)$ bits of memory.

Alternatively:

- Lower Bound: For error ϵ, must use at least $\Omega(1/\epsilon^{1/(2\ell-1)}c^{-2\ell+1})$
- Upper bound: For error ϵ, can solve the problem using at most $\tilde{O}(1/\epsilon^{4/\ell})$.

A communication problem

- Two players: Alice and Bob receive $a, b \in \{0,1\}^n$ respectively.
- Neither player knows the other's input.
- They want to compute $\mathrm{GT}_n(a,b) = 1$ if $a > b$, 0 otherwise. May pass r messages to each other to do so.
- Let $R^r(\mathrm{GT}_n)$ denote the size of the largest message that must be passed. Known: $R^r(\mathrm{GT}_n) = \Omega(n^{1/r}c^{1-r})$ [MNSW98].

Main Idea of Proving Lower Bound

- Assume there exists an ℓ-pass algorithm P that solves GLP with error ϵ^{ℓ} and uses $M(P)$ bits of memory.
- Show that above algorithm will induce a $2\ell - 1$ round protocol for communication game that solves GT problem for $n = 1/\epsilon^{\ell}$.
- Then show that $M(P) \geq R^{2\ell-1}(\mathrm{GT}_n) = \Omega(1/\epsilon^{1/2}c^{-2\ell+1})$.

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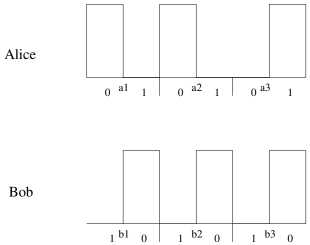

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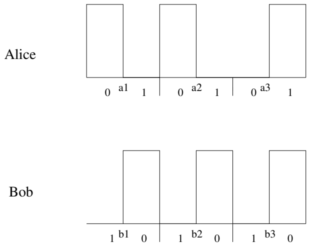

- They produce samples from their respective distribution and simulate a *single* input array for GLP. Alice and Bob each have one half of the input array.

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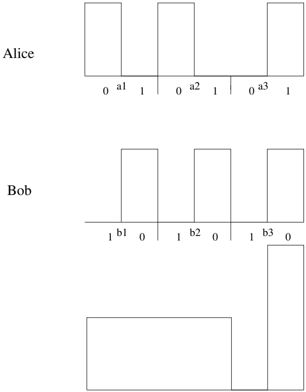
- By learning the first three steps of this distribution, they can deduce who has the larger number.

Example, continued

$a_1 a_2 a_3 = 001$ and $b_1 b_2 b_3 = 000$.



How to Simulate P by Passing Messages

- Technique from [AMS96].
- Alice simulates P on her half, sends P's memory to Bob. Bob simulates the rest of the pass on his half, and sends the memory back to Alice. Continue like this for ℓ passes.
- They pass a total of $2\ell - 1$ messages of size at most $M(P)$. Thus, $M(P) \geq R^{2\ell-1}(\mathrm{GT}_n)$.

Future Work

- Generalize the algorithm to learn mixtures of uniform distributions in dimension greater than 2.
- Design pass-efficient algorithms for learning mixtures of Normal Distributions in high dimension.
- *Heuristic* for learning mixture of 1-D normal distributions: Approximate mixture as k piecewise constant or uniform, and use our algorithm. See how this works empirically.

Thanks for listening!