Algorithms and Data Structures — **Summer 2008**
K. Mehlhorn and R. Seidel — **Tue, Oct. 7th**
Final Exam — **14:30 – 16:30**

1. (10 points) Let $G=(V,E)$ be an undirected graph, and assume that some of the edges in E are designated as "bad."

   How can you find a spanning tree of G that contains as few bad edges as possible?

   How fast can you construct such a spanning tree?

2. (20 points) In this question we consider hash functions from a key universe $U=\{0,\dots,t-1\}^d$ to table indices $\{0,\dots,t-1\}$. Here d is some fixed integer greater than 1 and t is a prime number.

   For each d-tuple $a=\langle a_1,\dots,a_d\rangle \in U$ define the function

   $$h_a(x)=\sum_{1\le i\le d} a_i x_i \bmod t\,,$$

   where $x=\langle x_1,\dots,x_d\rangle \in U$.

   In class we showed that the set

   $$\mathscr{H}_0=\{\,h_a;\,a\in U\,\}$$

   forms a universal set of hash functions.

   The random choice of a function from $\mathscr{H}_0$ requires to choose d random numbers $a_1,\dots,a_d$ from $\{0,\dots,t-1\}$.

   (a) Would it also suffice to choose just $d-1$ random numbers? In other words, is the set

   $$\mathscr{H}_1=\left\{h_a\,|\,a=\langle 1,a_2,\dots,a_d\rangle \in \{0,\dots,t-1\}^d\right\}$$

   universal?

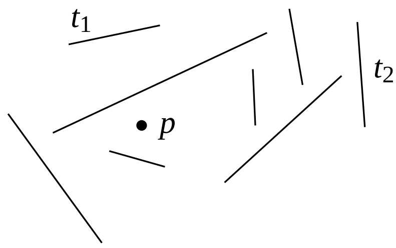
   (b) Would it also suffice to choose just $d-2$ random numbers? In other words, is the set

   $$\mathscr{H}_2=\left\{h_a\,|\,a=\langle 1,1,a_3,\dots,a_d\rangle \in \{0,\dots,t-1\}^d\right\}$$

   universal?

3. (15 points) We are given a set S of n nonintersecting line segments and a set P of n points. We want to find all pairs $(p,s)\in P\times S$ such that p lies on s. Design an algorithm and analyze its running time.

4. (20 points) Let S be a set of n disjoint line segments in the plane, and let p be a point that does not lie on any of the segments in S. We wish to determine all segments in S that p can see, i.e., all segments s that contain some point q so that the open line segments $\overline{pq}$ does not intersect any segment of S. Describe an $O(n \log n)$ algorithm that uses a rotating half-line with its endpoint at p. In the figure below, the segments t_1 and t_2 are not visible from p; all other segments are visible.



5. (15 points) Is the following statement true or false? If true, give a short explanation, if false, give a counterexample.

   Let $G = (V, E)$ be a directed graph, $s \in V$ a source vertex, $t \in V$ a sink vertex, and $cap : E \to \mathbb{N}$ a capacity function. Let (S, T) be a minimum s-t cut. Now suppose that we increase all capacities by one; then (S, T) is also a minimum s-t cut for the modified capacities.

6. (20 points) Let $G = (V, E)$ be a directed graph, $cap : E \to \mathbb{N}$ a capacity function, $c : E \to \mathbb{N}$ a cost function, and $b : V \to \mathbb{Z}$ a demand/supply function with $\sum_v b(v) = 0$.

   (a) Let $f : E \to \mathbb{N}$ be a flow. Describe how to check whether f is a minimum cost flow satisfying the demands and supplies. What is the running time of your method?

   (b) Argue that there is a minimum cost flow in which the flow across each edge is integral. Observe that all capacities and costs are integral.

   (c) Assume that f is a minimum cost flow. We increase the cost of one edge by one. Describe how to update f so that it becomes a minimum cost flow for the modified network. What is the cost of updating f?

7. (30 points) Let $G = (V, E)$ be a directed graph and let $s \in V$ be a designated node. For a subset $F \subseteq E$ of the edges, let $q(F)$ be the number of nodes that are not reachable from s in $(V, E \setminus F)$. We want to find the F that maximizes $g(F) := q(F) - |F|$; for $F = \emptyset$, we have $g(F) = 0$. Describe a polynomial time algorithm for finding such an F.

Cake and coffee are served at 16:45 on first floor of MPI building.