Inversion

The following text is Section VIII.6.2 of [Meh84]. All unresolved references are to this book.

We will now briefly discuss a second transformation which establishes a correspondence between lines (planes) and circles (spheres) in $\mathbb{R}^2$ ($\mathbb{R}^3$). It frequently allows us to transform problems about circles into seemingly simpler problems about lines.

Inversion in $\mathbb{R}^2$ ($\mathbb{R}^3$) is most easily described with respect to a polar (spherical) coordinate system. Let c be the origin of the coordinate system. Then inversion with center c maps a point p with polar coordinates (R, Φ) (spherical coordinates (R, Φ, Ψ)) into point $(1/R, \Phi)$ $((1/R, \Phi, \Psi))$, i.e., it inverts the distance from the origin. We collect important properties of the inversion map in the following Lemma.

Lemma 1

a) *For all p: $I(I(p)) = p$*

b) *Let C be a circle (sphere) which passes through the center c of the inversion. Then $I(C)$ is a line (plane) which does not pass through c.*

c) *Let P be a line (plane) which does not pass through c. Then $I(P)$ is a circle (sphere) which passes through c.*

Proof: a) is obvious.

b)and c): We only give the proof for $\mathbb{R}^2$ and leave the three-dimensional case to the reader. We first express inversion in Cartesian coordinates.

Lemma 2 *Let (x, y) be the Cartesian coordinates of point p with respect to origin c. Then $I(p)$ has Cartesian coordinates $(x/(x^2 + y^2), y/(x^2 + y^2))$.*

Proof: Let (R, Φ) be the polar coordinates of point p. Then $R^2 = x^2+y^2$ and $\tan \Phi = y/x$. Let $\overline{x} = x/(x^2 + y^2)$ and $\overline{y} = y/(x^2 + y^2)$. Then $\tan \Phi = \overline{y}/\overline{x}$ and $\overline{x}^2 + \overline{y}^2 = 1/R^2$. Hence $I(p)$ has Cartesian coordinates $(\overline{x}, \overline{y})$. ∎

We are now in a position to prove part c). Part b) is proven similarly and is left to the reader. Because of symmetry it suffices to prove part c) for a horizontal line.

Let h be a horizontal line given by the equation $y = a$. Let p with Cartesian coordinates (x, a) be an arbitrary point on line h. Then $I(p)$ has Cartesian coordinates $(x/d, a/d)$, where $d = x^2 + y^2$. It is now easy to check that $I(p)$ lies on the circle with center $(0, 1/2a)$ and radius $1/2|a|$. Also, all points on this circle are images of points on line h. ∎

Our first application of the inversion transform directly uses Lemma 1. Let C_i, $1 \leq i \leq n$, be a circle in $\mathbb{R}^2$ and let $s_i \in \{+, -\}$, $1 \leq i \leq n$. We use $C_i^{s_i}$ to denote the interior (exterior) of circle C_i provided that $s_i = +$ ($s_i = -$).

Lemma 3 *If $C_1, \dots, C_n$ are circles in $\mathbb{R}^2$ which all pass through point c and $s_i \in \{+,-\}$, $1 \le i \le n$, then*

$$\bigcap_{i=1}^{n} C_i^{s_i}$$

can be computed in time $O(n \log n)$.

Proof: Let I be the inversion with center c. Let $h_i = I(C_i)$, $1 \le i \le n$. Then h_i is a line which does not pass through c and $C_i^{s_i}$ corresponds to one of the halfspaces defined by h_i. We denote that halfspace by $h_i^{s_i}$. Let $P = \bigcap_i h_i^{s_i}$ be the convex polygon defined by the intersection of the halfspaces $h_i^{s_i}$, $1 \le i \le n$. We can compute P in time $O(n \log n)$ by Theorem **??**. Also, $I(P) = \bigcap_i C_i^{s_i}$. Furthermore,it is easy to compute $I(P)$ since vertices of P are transformed into points and edges are transformed into "circular edges". Thus $I(P)$ can be computed in time $O(n)$ and the proof is complete. ∎

The preceding Lemma hinges on the artificial assumption that all circles have a common point c. This assumption can be replaced if we invest one more idea: embedding into higher dimension.

Theorem 1 *Let C_i be circles in $\mathbb{R}^2$ and let $s_i \in \{+,-\}$, $1 \le i \le n$. Then $\bigcap_i C_i^{s_i}$ can be computed in time $O(n \log n)$.*

Proof: Identify $\mathbb{R}^2$ with the xy-plane of $\mathbb{R}^3$. Let c be a point outside the xy-plane, say $c = (0,0,1)$ and let S_i be a sphere which passes through c and intersects the xy-plane in circle C_i, $1 \le i \le n$. Let $h_i = I(S_i)$, $1 \le i \le n$, be the plane obtained by inversion of S_i with respect to center c. Then $S_i^{s_i}$ corresponds to one of the half-spaces defined by plane h_i. We use $h_i^{s_i}$ to denote that half-space. Then convex polyhedron $P = \bigcap_i h_i^{s_i}$ can be computed in time $O(n \log n)$ by Theorem **??**. Also, $I(P) = \bigcap_i S_i^{s_i}$ and hence the intersection of $I(P)$ with xy-plane is the desired solution. We can compute $I(P) \cap (xy\text{-plane})$ in time $O(n)$ by transforming face by face of $I(P)$. The details are left to the reader.

Let me give a few more details (16 years later). The boundary of P consists of vertices, edges (= straight-line segments), and faces (plane patches). Inversion maps the vertices into points in $\mathbb{R}^3$, the edges into circular arcs connecting these points, and the faces into patches on our spheres S_i. Intersecting these features with the xy-plane gives us $\bigcap_i C_i^{s_i}$. The intersection of a point with the xy-plane is either empty or the point itself[1], the intersection of a circular arc with the xy-plane is either empty or a point or the circular arc itself (the latter situation cannot happen, because it would require an edge of P to be contained in $I(xy\text{-plane})$ which is a sphere), and the intersection of a spherical patch with the xy-plane is either empty or a circular arc.

We conclude that the vertices of $\bigcap_i C_i^{s_i}$ are either preimages of vertices of P or intersections of preimages of edges of P with the xy-plane and that the edges of $\bigcap_i C_i^{s_i}$ are intersections of preimages of faces of P with the xy-plane. ∎

[1]Such a point must then lie in the xy-plane and on three of our input circles, i.e., it corresponds to a degenerate situation.

I see two ways of actually implementing the theorem above.

1. One either follows the theorem literally, i.e., one constructs the spheres S_i, inverts them, computes the intersection of halfspaces, and translates back to the original situation.

2. One takes the theorem as a hint that the algorithmic strategies that work for the intersection of halfspaces also work for the intersection of circular regions. One develops an algorithm that works directly for circles and implements it. The randomized incremental algorithm for halfspace intersection is a good candidate.

3. In either approach one has to work out the geometric primitives. For the second approach one needs at least the following. Given three circles C_i, $1 \le i \le 3$, which intersection points of C_1 and C_2 are contained in the interior, lie on the boundary, are outside C_3. This is similar to the primitives that Susan Hert discussed last time.

The approach also works in higher dimensions, i.e., the intersection of spheres in $\mathbb{R}^3$ translates to intersection of halfspaces in $\mathbb{R}^4$.

Algorithms for intersection problems on circles and spheres would make a very nice Diploma thesis.

Theorem 1 deals with circles in a very direct way. In Voronoi diagrams circles come up in a more subtle way. Let $S = \{p_1, \ldots, p_n\}$ be a set of n points in the plane and let VD be the Voronoi diagram of S; cf. Section 3.1. Let v be any vertex of the Voronoi diagram. Then v is the center of a circle $C(v)$ which passes through at least three points of S and has no point of S in its interior. Conversely, the center of any such circle is a vertex of the diagram. As in Theorem 1, we identify $\mathbb{R}^2$ with the xy-plane of $\mathbb{R}^3$, choose a point c outside the xy-plane and use $S(v)$ to denote a sphere which passes through c and intersects the xy-plane in circle $C(v)$. Consider the inversion I with center c. Let $h(v) = I(S(v))$. Then $h(v)$ is a plane such that at least three points in $I(S)$ lie on $h(v)$ and all other points in $I(S)$ belong to the same half-space with respect to $h(v)$. In other words, plane $h(v)$ supports a face of the convex hull of point set S.

Construct $I(S)$ and compute the convex hull of $I(S)$. This takes time $O(n \log n)$ by Exercise 13. Let f be an arbitrary face of $I(S)$, let E be the supporting plane, and let $SP = I(E)$ be he sphere obtained by inversion of E. Then all points of S lie either inside or on SP or all of them lie outside or on SP. The two cases are easily distinguished by testing one point of S (whose image under I does not lie on face f) with respect to SP. In the latter case the center of the circle obtained by intersecting SP with the xy-plane is a vertex of the Voronoi diagram (in the former case it is a vertex of the farthest point Voronoi diagram). In this way we compute all vertices of the Voronoi diagram in time $O(n)$, each vertex of the diagram corresponding to a face of the convex hull of $I(S)$. Now the edges are quickly computed. We connect vertices v and w by a (straight-line) edge if the corresponding faces of the convex hull share an edge. This proves

Theorem 2 *The Voronoi diagram of a point set $S \subseteq \mathbb{R}^2$, $|S| = n$, can be computed in time $O(n \log n)$.*

References

[Meh84] K. Mehlhorn. *Data Structures and Algorithms 3: Multidimensional Searching and Computational Geometry.* Springer, 1984.