1 The Ellipsoid Method

We start with a one-dimensional problem. Consider the following situation.

- The goal is to find an $x \in \mathbb{R}$ having a certain property P or to tell that no $x \in R$ has property P.
- We know that the set S of x having property P is either empty or an interval of length at least ℓ.
- We know that the absolute value of any x with property P is bounded by U.
- Given an $x \in \mathbb{R}$ we can test whether x has property P. If x does not have property P, either $S \subseteq [-\infty \mathinner{..} x]$ or $S \subseteq [x \mathinner{..} +\infty]$. This follows from the fact that S is an interval. We can tell which of the two cases applies[1].

The problem just described can be solved by binary search.

1. Initialize an interval I to $[-U \mathinner{..} U]$. Then $S \subseteq I$. In the course of the algorithm we will shrink I and maintain the invariant that $S \subseteq I$.
2. If the length of I is less than ℓ, we stop and declare that there is no x with property P. This is correct, since $S \subseteq I$ by our invariant. Moreover, if S is non-empty its length is at least ℓ. Thus if the length of I is less than ℓ, S must be empty.
3. So assume that the length of I is larger than ℓ. Let z be the midpoint of I. If z has property P, we stop.
4. Otherwise, we replace I by either the left half or the right half of I and continue with step 2. We exclude the half which is known to contain no point in I and hence the invariant $S \subseteq I$ is maintained.

We have already argued correctness of our method. Let us next bound the number of iterations. After k executions of step four the length of I is $2^{-k}2U$, since we start with an interval of length $2U$ and since every execution of step four halves the length of the interval.

Assume that the loop body is executed $k+2$ times, i.e., in the $k+2$-nd iteration we stop in either step 2 or step 3. Since we did not stop in iteration $k+1$, the length of I at the beginning of the $k+1$-st iteration (=end of k-th iteration) is at least ℓ. Thus $2^{-k}(2U) \geq \ell$ or $k \leq \log((2U)/\ell)$.

Theorem 1 *Binary search solves the search problem described above with at most*

$$2 + \log \frac{2U}{\ell}$$

iterations.

[1] This item is usually called a *separation oracle*: the word oracle emphasizes that, at least for the purposes of the current discussion, it is irrelevant how the decision is made, separation indicates that the decision separates z from S.

Proof: by the preceding discussion. ∎

Before we generalize to higher dimensions, we observe that it is not really important that the length of I is halved in every iteration. Any reduction by a constant factor $f < 1$ would do. Assume the length of I is reduced by a factor $f < 1$ in every iteration. Then the last paragraph has to be changed to:

Assume that the loop body is executed $k+2$ times, i.e., in the $k+2$-nd iteration we stop in either step 2 or step 3. Since we did not stop in iteration $k+1$, the length of I at the beginning of the $k+1$-st iteration (=end of k-th iteration) is at least ℓ. Thus $f^k(2U) \geq \ell$ or $k \leq \log_{1/f}((2U)/\ell) = \frac{\log((2U)/\ell)}{\log(1/f)}$. Thus only the basis of the log-function changes.

The ellipsoid method generalizes binary search to higher dimension. We discuss the generalization to two dimensions. Let us first generalize our search problem. Consider the following situation.

- The goal is to find an $x \in I\!R^2$ having a certain property P or to tell that no $x \in R^2$ has property P.
- (Minimum Area) We know that the set S of x having property P is either empty or a convex set of area at least ℓ.
- (Localization) We know that any x with property P is contained in a disk with radius U centered at the origin.
- (Separation Oracle) Given an $x \in I\!R^2$ we can test whether x has property P. If x does not have property P, there is a line l passing through x such one of the (open) half-spaces defined by l contains no point in S. This follows from the fact that S is convex. We assume that we can determine such a line and the empty open half-space (equivalently, the closed half-space H with boundary l and containing S)

How can we generalize binary search? What is the proper generalization of the interval I which is known to contain S? Khachian showed that ellipses work. This leads to the following algorithm.

1. Initialize E to the disk with radius U centered at the origin. Then $S \subseteq E$. In the course of the algorithm we will shrink E and maintain the invariants that
   - E is an ellipse
   - E contains S.
2. If the area of E is less than ℓ, we stop and declare that there is no x with property P. This is correct, since $S \subseteq I$ by our invariant. Moreover, if S is non-empty its area is at least ℓ. Thus if the area of E is less than ℓ, S must be empty.
3. So assume that the area of E is larger than ℓ. Let z be the center of E. If z has property P, we stop.

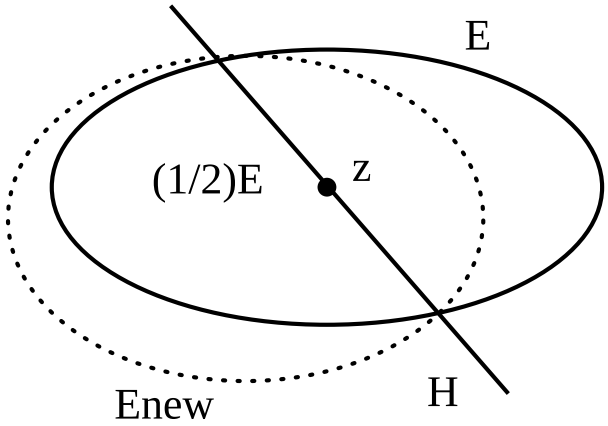

Figure 1: The figure shows an ellipse E, and a halfspace H having z, the center of E, on its boundary; $(1/2)E$ is the intersection of E and H. *Enew* is an ellipsoid containing $(1/2)E$. It is not the smallest such ellipsoid (my mastering of xfig did not suffice for this purpose).

4. Otherwise, by our assumption, we can determine a closed half-plane H having z in its boundary and containing S. Define $(1/2)E$ as the intersection of E and H. Then $S \subseteq (1/2)E$ and the area of $(1/2)E$ is one-half the area of E.

   Unfortunately, $(1/2)E$ is not an ellipsoid. Here is where the ellipsoid method goes beyond binary search. In binary search $(1/2)I$ is an interval and hence we immediately proceed to the next iteration. In the ellipsoid method we need one further step.

   Set E to the smallest (in area) ellipse containing $(1/2)E$, see Figure 1.

Clearly, if the method terminates, it terminates with the correct answer. The key for the termination proof and the running time analysis is to show that the area of E is multiplied by a factor f less than one in every iteration.

How can we determine the smallest ellipsoid containing $(1/2)E$? Let us start with a particularly simple situation, see Figure 2: E is a unit disk centered at the origin and H is the left half-plane.

The figure suggests to use an ellipse E' that has its center on the negative x-axis, passes through points $(-1,0)$, $(0,1)$ and $(0,-1)$, and has axes parallel to the coordinate axes. Thus

$$E' = \left\{(x,y) \in R^2 \; ; \; ((x-c)/a)^2 + (y/b)^2 \leq 1\right\}$$

for appropriate constants a, b, and c. We have $-1 \leq c \leq 0$ and a and b are the length of the axes of our ellipse, $a > 0$ and $b > 0$. The area of E' is πab and we are going to choose a and b such that the area is minimal. Since $(-1,0)$ lies on the boundary of E', the length of the horizontal axis is $1+c$. Thus $a = 1+c$. We must clearly have $2a \geq 1$. Thus $-1/2 \leq c \leq 0$. Since $(0,1)$ lies on the boundary of E' we have $(c/(1+c))^2 + (1/b)^2 = 1$.

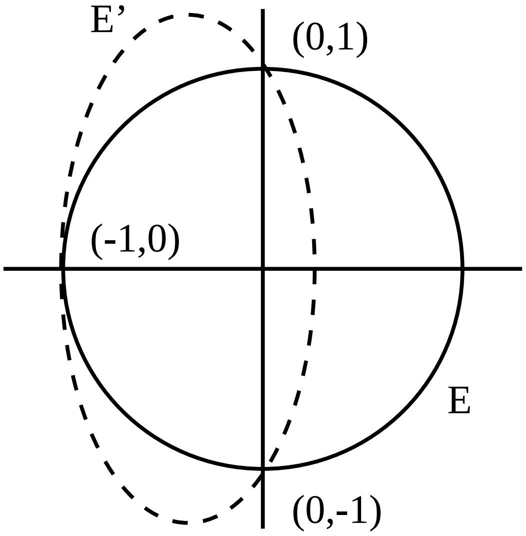

Figure 2: E is the unit disk and H is the left half-plane. The figure suggests that the smallest ellipse E' containing $E \cap H$ has its center on the negative x-axis, and passes through points $(-1,0)$, $(0,1)$ and $(0,-1)$, and has axes parallel to the coordinate axes.

Thus $(1/b)^2 = 1 - (c/(1+c))^2 = ((1+c)^2 - c^2)/(1+c)^2$ and hence $b = (1+c)/\sqrt{1+2c}$. At this point, we are left with a single parameter c. For any c with $-1/2 < c \le 0$, the formulae above determine a and b.

The area of E' is $\pi ab = \pi(1+c)^2/\sqrt{1+2c}$. For $c = 0$, E' is equal to E and the area is equal to π, the area of the unit disk. For $c = -0.1$, the area of E' is $\pi 0.9^2/\sqrt{0.8} \approx 0.9\pi$ and hence the area of E' is only 90% of the area of E. We can do even better, but there is actually no need to do so. Already at this point, we know that the number of iterations is logarithmic in U/ℓ.

The area is minimized[2] for $c = -1/3$. Then

$$\frac{area(E')}{area(E)} = \frac{(2/3)^2}{\sqrt{1/3}} = \frac{4\sqrt{3}}{9} \le 0.8 \ .$$

The center of E' is at $(-1/3, 0)$ and the axes have length $2/3$ and $2/\sqrt{3}$ respectively. Thus $1/a^2 = 9/4$ and $1/b^2 = 3/4$.

We still need to verify that E' contains $E \cap H$. So let (x,y) be arbitrary with $x^2 + y^2 \le 1$ and $x \le 0$. Then

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = \frac{9}{4}(x+\frac{1}{3})^2 + \frac{3}{4}y^2 = \frac{6}{4}x^2 + \frac{3}{2}x + \frac{1}{4} + \frac{3}{4}(x^2+y^2) \le \frac{3}{2}x^2 + \frac{3}{2}x + 1 \le 1 \ .$$

[2]The derivative of $(1+c)^2/\sqrt{(1+2c)}$ with respect to c is $2(1+c)(1+c)^{-1/2} - (1+c)^2(1+2c)^{-3/2} = (2(1+c)(1+2c) - (1+c)^2)(1+2c)^{-3/2} = (1+c)(2(1+2c) - (1+c))(1+2c)^{-3/2}$. We have maxima for $c = -1$ (an illegal value) and $2 + 4c - 1 - c = 0$ or $3c = -1$ or $c = -1/3$.

The next to last inequality follows from $x^2 + y^2 \leq 1$ and the last inequality follows from $x^2 + x \leq 0$ for $-1 \leq x \leq 0$.

How can we deal with the case of an arbitrary ellipse E. We can derive the equations for E' as follows:

1. translate space so that z is moved into the origin.
2. rotate the plane such that the axes of E become aligned with the coordinate axes.
3. scale x and y-coordinates such that E becomes the unit disk
4. at this point the boundary of H is arbitrary line through the origin.
5. rotate space again so as to turn H into the left half-plane.
6. at this point we are in our special situation and hence know E'.
7. apply steps 1 to 5 in reverse order to E'.

A nice fact about the transformations used in steps 1 to 5 is that, although they change volumes they do not change the ratio of volumes and hence the bound on the ratio of the volume of E and E' derived above stays true.

Theorem 2 *The ellipse method solved the two-dimensional search problem with at most*

$$2 + \log_{1/f} \frac{4U^2}{\ell} = 2 + \frac{\log(4U^2/\ell)}{\log(9/(4\sqrt{3}))}$$

iterations; $f = 4\sqrt{3}/9$.

Proof: We start with the disk of radius U centered at the origin. Its area is $\pi U^2 \leq 4U^2$. We terminate when the area is smaller than ℓ and we reduce the area by a factor $f = 4\sqrt{3}/9$ in every iteration. Therefore the number of iterations is at most ∎

I hope it is clear now. If not, I tried my best.