max planck institut informatik

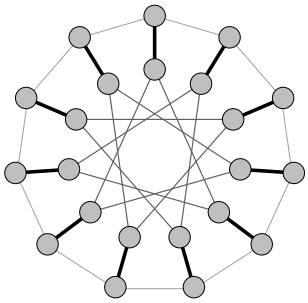
Selected Topics in Algorithms **Summer 2009**
K. Mehlhorn **week starting May 4th**
Exercise 1

Motivation

We want to deepen our understanding of cycle bases.

Undirected versus Integral Cycle Bases

In class, we exhibited a weighted graph for which the minimum directed cycle basis has smaller weight than the minimum undirected cycle basis. The goal of this exercise is to construct a graph and a weight function for which the minimum undirected cycle basis has smaller weight than the minimum integral cycle basis. Consider the Petersen graph $P_{11,4}$. It has vertices I_i (inner vertices) and O_i (outer vertices) for $0 \leq i \leq 10$ and edges I_iO_i (spokes), O_iO_{i+1} (outer edges), and I_iI_{i+4} (inner edges) for $0 \leq i < 10$. All indices are modulo 11.



1. How many circuits are contained in a cycle basis of $P_{11,4}$?
2. Set the weight of the spokes to a, the weight of the outer edges to b, and the weight of the inner edges to c.
   - There are two circuits that use no spoke. What is their weight?
   - What is the minimum weight of a circuit containing at least four spokes.
   - Classify the circuits using exactly 2 spokes according to the number of outer edges used. How many inner edges do they use.
   - Choose the weights such that the outer circuit together with the 11 circuits using one outer and three inner edges form the unique minimum undirected cycle basis.
   - Argue that this basis is not integral.

Exchange Theorem

Let k be a field, let B be a k-basis, let $C \in B$, and let $C = C_1 + C_2$. Then either $B \setminus \{C\} \cup \{C_1\}$ or $B \setminus \{C\} \cup \{C_2\}$ is also a k-basis.

Simplification Rules

Let k be a field and $G = (V,E)$ be a graph with weight function w.

- Let $e \in E$ and let C be a minimum weight circuit containing e. Then there is a minimum weight k-basis containing e.
- Let e_1 and e_2 be parallel edges with $w(e_1) \leq w(e_2)$. Then the following constitutes a minimum weight k-basis of G: a minimum weight k-basis of $G \setminus e_2$ plus a minimum weight circuit containing e_2.
- Let e be an edge with $w(e) = 0$ and no edge parallel to it. Relate minimum weight k-bases of G and G', where G' is obtained from G by contracting e.

Short Circuits in Graphs

Let $k \in \mathbb{N}$ and $G = (V,E)$ be an undirected graph with minimum degree $n^{1/k}$. Show that G contains a circuit of length $O(k)$.

Have fun with the solution!