Selected Topics in Algorithms
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Exercise 2

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We will discuss this exercise sheet on May 15^{th} and May 18^{th}

Motivation

We continue our study of cycle bases.

Small Weight Cycle Bases in Complete Graphs

Let K_n be the complete graph on n nodes. In K_n, any two nodes are connected by an edge. Let $w : E \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative weight function on the edges and let $W = \sum_{e \in E} w(e)$ be the total weight of all edges. Show that there is a cycle basis of weight $O(W)$. Design an algorithm to find such a basis. What is its running time?
Hint: For any node v consider the spanning tree T_v consisting of all edges incident to v. Let B_v be the fundamental basis with respect to this spanning tree.

Shortest Paths in Undirected Graphs

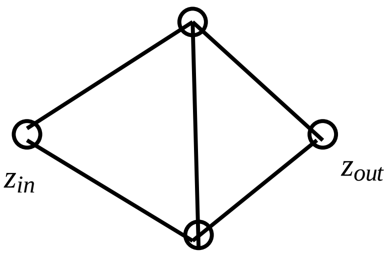
Let $G = (V,E)$ be an undirected graph and let $w : E \longrightarrow \mathbb{R}_{\geq 0}$ be a weight function on the edges. We allow negative weights.
For nodes u and v, let $d(u,v)$ be the minimal weight of a simple path from u to v. Recall that a path $(v_0, v_1, \ldots, v_k)$ is simple, if $v_i \neq v_j$ for $i \neq j$ and $(i, j) \neq (0,k)$. The latter exception allows for cycles.

NP-Completeness: Show that the shortest simple path problem is NP-complete. As part of this exercise you should first give a precise definition of the shortest simple path problem. Hint: consider the case $w(e) = -1$ for all e.

Conservative Weights: We call w *conservative* if $w(C) \geq 0$ for any simple cycle C. Show how to how to solve the shortest simple path problem with respect to a conservative weight function in polynomial time.

Hint 1: you may use the fact that the minimum weight perfect matching problem is in P. This problem asks to compute a subset $M \subseteq E$ such that for any node v exactly one incident edge is in M, i.e., M is a perfect matching, and $w(M)$ is minimal.

Hint 2: In order to find a shortest simple path connecting u and v, construct an auxiliary graph as follows: For any node z distinct from u and v put the following gadget into the graph; you must invent similar gadgets for u and v. All edges in the gadget have weight zero. For any edge $e = (y,z) \in E$ add the edges (y_{out}, z_{in}) and (z_{out}, y_{in}) to the auxiliary graph. The weight of these edges is $w(e)$.



Show:

- for any simple path p in G connecting u and v, there is a perfect matching M in the auxiliary graph with $w(M) \leq w(p)$.
- for any perfect matching M in the auxiliary graph, there is a simple path in G connecting u and v with $w(p) \leq w(M)$.

Consistent Shortest Paths

We assume the weight function to be non-negative. For nodes u and v, let p_{uv} be a shortest path from u to v. We call a system $S = \{\, p_{uv};\, u,v \in V \,\}$ of shortest paths consistent, if for any $p_{uv} \in S$ and any two nodes y and z on p_{uv}, the subpath of p_{uv} connecting y and z is equal to p_{yz}.

- Give an example of a graph and a system of shortest paths that is not consistent.
- Show: if shortest paths are unique, then the system

  $$\{\, p_{uv};\, u,v \in \text{ and } p_{uv} \text{ is the unique shortest path from } u \text{ to } v \,\}$$

  is consistent.
- Show how to modify the edge weigths such that (1) shortest paths are unique under the modified weight function and (2) shortest paths under the modified weight function are shortest paths under the original weight function.
- Does the solution to the previous exercise lead to an efficient algorithm for computing a consistent set of shortest paths.
- Design an algorithm for computing a consistent set of shortest paths.

Have fun with the solution!