Approximate Maximum Flow in Undirected Networks by Christiano, Kelner, Madry, Spielmann, Teng (STOC 2011)

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The Result

- $G = (V, E)$ undirected graph, $s$ source, $t$ sink.
- $u : E \to \mathbb{R}_{\geq 0}$, edge capacities
- $\epsilon > 0$

Can compute $(1 - \epsilon)$-approximate maximum flow in time $\tilde{O}(mn^{1/3}\epsilon^{-11/3})$.

- approximate minimum cut in similar time bound
- previous best: $\tilde{O}(m\sqrt{n}\epsilon^{-1})$ by Goldberg and Rao (98)
- uses electrical flows
Conversion to Integral Capacities

- \( B = \max_{P \text{ is } s-t \text{ path}} \min_{e \in P} u_e \)
- max bottleneck path in time \( O(m + n \log n) \)
- \( B \leq \text{max flow} \leq mB \).
- replace \( u_e \) by \( \min(u_e, mB) \).
- removing all edges of capacity less than \( \epsilon B/(2m) \) changes max-flow by at most \( \epsilon B/2 \).
- replace \( u_e \) by \( \left\lfloor \frac{u_e}{\epsilon B/2m} \right\rfloor \)

integral capacities in \([1, 2m^2/\epsilon]\)
A High-Level View of the Algorithm

\( F^* = \text{value of maximum flow.} \)

Do binary search on \([1, 2m^2/\epsilon]\). Let \( F \) be the current value of the search.

Have a subroutine \( \text{Flow}(F) \) which

- either finds a flow of value \( F \) that almost satisfies the capacity constraints or fails.
- if \( F \leq F^* \), it is guaranteed to return a flow.

Subroutine is realized via a low-level subroutine \( \text{flow}(F, w) \), which we discuss first. Here, \( w \) is a weight function on the edges.
Electrical Flows and Capacities

Resistances can simulate capacities

Let $Q^*$ be a maximum flow. Orient edges in the direction of the flow, sort the graph topologically, and set

$$\rho_v = \text{number of nodes after } v \text{ in ordering.}$$

For $e = (u, v)$, set

$$R_e = \frac{Q^*_e}{\Delta_e}.$$  

Then $Q^*$ is the resulting flow.
An Observation

Set $R_e = 1/u_e^2$ and let $F \leq F^*$. Let $Q$ be an electrical flow of value $F$. Then

$$\sum_e (Q_e/u_e)^2 = \sum_e R_e Q_e^2 \leq \sum_e R_e (Q^*_e)^2 = \sum_e (Q^*_e/u_e)^2 \leq m.$$ 

Define the congestion of $e$ as

$$\text{cong}_e := Q_e/u_e.$$ 

Then,

$$\frac{1}{m} \sum_e \text{cong}_e^2 \leq 1 \quad \text{and} \quad \max_e \text{cong}_e \leq \sqrt{m}.$$
The Subroutine flow\((F, w)\)

1. Set \(R_e = \left( w_e + \epsilon W/m \right)/u_e^2 \), where \( W = \sum_e w_e \).
2. Let \( Q \) be an electrical flow of value \( F \).
3. If \( \sum_e R_e Q_e^2 > (1 + \epsilon) W \) declare failure.
4. return \( Q \).

Properties

If \( F \leq F^* \), flow does not fail

If flow succeeds,

\[
\sum_e \frac{w_e}{W} \text{cong}_e \leq 1 + \epsilon \quad \text{and} \quad \max_e \text{cong}_e \leq \rho := \sqrt{\frac{(1 + \epsilon)m}{\epsilon}}.
\]
Proof

Set $R_e = (w_e + \epsilon W/m)/u_e^2$, where $W = \sum_e w_e$. Let $Q$ be an electrical flow of value $F$. If $F \leq F^*$ then

$$\sum_e R_e Q_e^2 \leq \sum_e R_e (Q_e^*)^2 = \sum_e (w_e + \epsilon W/m) \left( \frac{Q_e^*}{u_e} \right)^2 \leq (1 + \epsilon) W.$$ 

If

$$\sum_e (w_e + \epsilon W/m) \left( \frac{Q_e}{u_e} \right)^2 = \sum_e R_e Q_e^2 \leq (1 + \epsilon) W$$

then

$$\sum_e \frac{w_e}{W} \text{cong}_e^2 \leq 1 + \epsilon \quad \text{and} \quad \max_e \text{cong}_e \leq \sqrt{\frac{(1 + \epsilon)m}{\epsilon}}.$$
From average squared congestion to average congestion

\[ \sum_e w_e \text{cong}_e = \sum_e w_e^{1/2} \cdot w_e^{1/2} \text{cong}_e \]

\[ \leq \left( \sum_e w_e \right)^{1/2} \cdot \left( \sum_e w_e \text{cong}^2_e \right)^{1/2} \]

\[ \leq W^{1/2} ((1 + \epsilon) W)^{1/2} \leq (1 + \epsilon) W. \]
From flow to Flow

Flow($F$)

- set $w_e^{(1)} = 1$ for all $e$;
- for $i = 1 \rightarrow T$ do
  - $Q^{(i)} = \text{flow}(F, w)$;
  - $\text{cong}_e^{(i)} = Q_e^{(i)}/u_e$ for all $e$
  - $w_e^{(i+1)} = w_e^{(i)} (1 + \epsilon\text{cong}_e^{(i)}/\rho)$ for all $e$;
- end for
- return

\[ Q := \frac{1}{T} \sum_{1 \leq i \leq T} Q^{(i)} \]

\[ \{ T = O(m^{1/2}\epsilon^{-5/2}) \text{ suffices} \} \]
\[ \{ \text{if call fails, fail} \} \]
Properties of Flow

Q is a flow of value $F$ and if $F \leq F^*$, $Q$ exists.

$$Q_e = \frac{1}{T} \sum_{1 \leq i \leq T} Q^{(i)}_e = \frac{1}{T} \sum_{1 \leq i \leq T} u_e \cdot \text{cong}^{(i)}_e = u_e \cdot \overline{\text{cong}}_e$$

$$W^{(i+1)} = \sum_e w^{(i)}_e \left(1 + \epsilon \text{cong}^{(i)}_e / \rho\right) \leq \left(1 + \epsilon (1 + \epsilon) / \rho\right) W^{(i)}$$

$$W^{(T+1)} \leq \exp(((1 + \epsilon)\epsilon / \rho) T) \cdot m$$
Properties of Flow

\[ w_e^{(i+1)} = w_e^{(i)} (1 + \epsilon \text{cong}_e^{(i)}/\rho) \geq w^{(i)} \exp((1 - \epsilon)\epsilon \text{cong}_e^{(i)}/\rho) \]

\[ w_e^{(T+1)} \geq \exp((1 - \epsilon)\epsilon \text{cong}_e/\rho) T \]

\[ ((1 - \epsilon)\epsilon \text{cong}_e/\rho) T \leq \ln m + (\epsilon(1 + \epsilon)/\rho) T \]

\[ \text{cong}_e \leq \frac{\rho \ln m}{(1 - \epsilon)\epsilon T} + \frac{1 + \epsilon}{1 - \epsilon} \leq \frac{\epsilon}{(1 - \epsilon)} + \frac{1 + \epsilon}{1 - \epsilon} \leq 1 + 4\epsilon \]

for \( T = (\rho \ln m)/\epsilon^2 = \tilde{O}(m^{1/2} \epsilon^{-5/2}) \)
Putting it together

\[ \tilde{O}(m^{1/2} \epsilon^{-5/2}) \] iterations suffice.

In each iteration we need to solve a SSD system and do linear extra work. Thus an iteration runs in time \( \tilde{O}(m \log 1/\epsilon) \).

Total running time is \( \tilde{O}(m^{3/2} \epsilon^{-5/2}) \).

But, I promised \( \tilde{O}(mn^{1/3} \epsilon^{-11/3}) \). This is reached in two steps:

- step one reduces to \( \tilde{O}(m^{4/3} \epsilon^{-3}) \), and
- step two reduces to \( \tilde{O}(mn^{1/3} \epsilon^{-11/3}) \). (Karger (98) and Bencur/Karger (02))
The First Step

Let $H$ be a huge number; actually $H = (m \ln m)^{1/3} / \epsilon$.

What does $\text{cong}_e \geq H$ imply?

$Q_e / u_e \geq H$ and hence $u_e \leq Q_e / H \leq F / H$. Thus $u_e$ is tiny.

We can afford to delete $\epsilon H$ edges with huge congestion without sacrificing the approximation guarantee.

Modification of flow: if flow succeeds, i.e., $\mathcal{E}(Q) \leq (1 + \epsilon) \mathcal{W}$, and there is an edge $e$ with huge congestion, delete the edge and continue without the edge.

Observe, that change allows us to replace $\rho$ by $H$ in the analysis.
Deleting Huge Edges

If flow succeeds, we have $\mathcal{E}(Q) \leq (1 + \epsilon)W$.

If $e$ has huge congestion,

$$r_e Q_e^2 \geq \frac{\epsilon W}{m} \left( \frac{Q_e}{u_e} \right)^2 \geq \frac{\epsilon H^2}{(1 + \epsilon)m} (1 + \epsilon)W \geq \frac{\epsilon H^2}{(1 + \epsilon)m} \mathcal{E}(Q).$$

Let $\beta = \epsilon H^2 / ((1 + \epsilon)m)$. If $e$ has huge congestion, $e$ accounts for a $\beta$ fraction of the energy of the flow.

Deletion of a huge edge forces the energy of the flow to increase by a factor $1/(1 - \beta)$.

We have an upper bound on the final energy, namely $(1 + \epsilon)W^{(T+1)}$. It is not too hard, to derive a lower bound on the energy of the first flow. Putting things together, we obtain a bound on the number of huge edges.
Deleting a Huge Edge II

Deletion of a huge edge increases the energy of the flow by a factor $1/(1 - \beta)$.

Let $p$ be the electrical potentials for flow of value $1/R_{eff}$. Then $p_s = 1$ and $p_t = 0$. Energy of this flow is equal to $1/R_{eff}$.

$$\frac{1}{R'_{eff}} = \inf_{\substack{q_s = 1 \quad q_t = 0 \quad uv \in E \setminus e \quad (q_u - q_v)^2 \quad r_{uv}}} \leq \sum_{uv \in E \setminus e} \frac{(p_u - p_v)^2}{r_{uv}}$$

$$= \sum_{uv \in E} \frac{(p_u - p_v)^2}{r_{uv}} - \Delta_{e}^2/r_{e} \leq (1 - \beta) \frac{1}{R_{eff}}$$

Thus, $\mathcal{E}(Q') = F^2 R'_{eff} \geq \frac{1}{1 - \beta} F^2 R_{eff} = \mathcal{E}(Q)$. 