Curve and Surface Reconstruction

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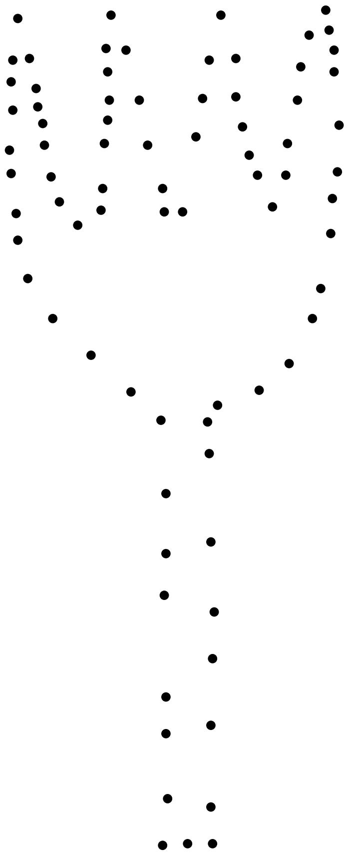
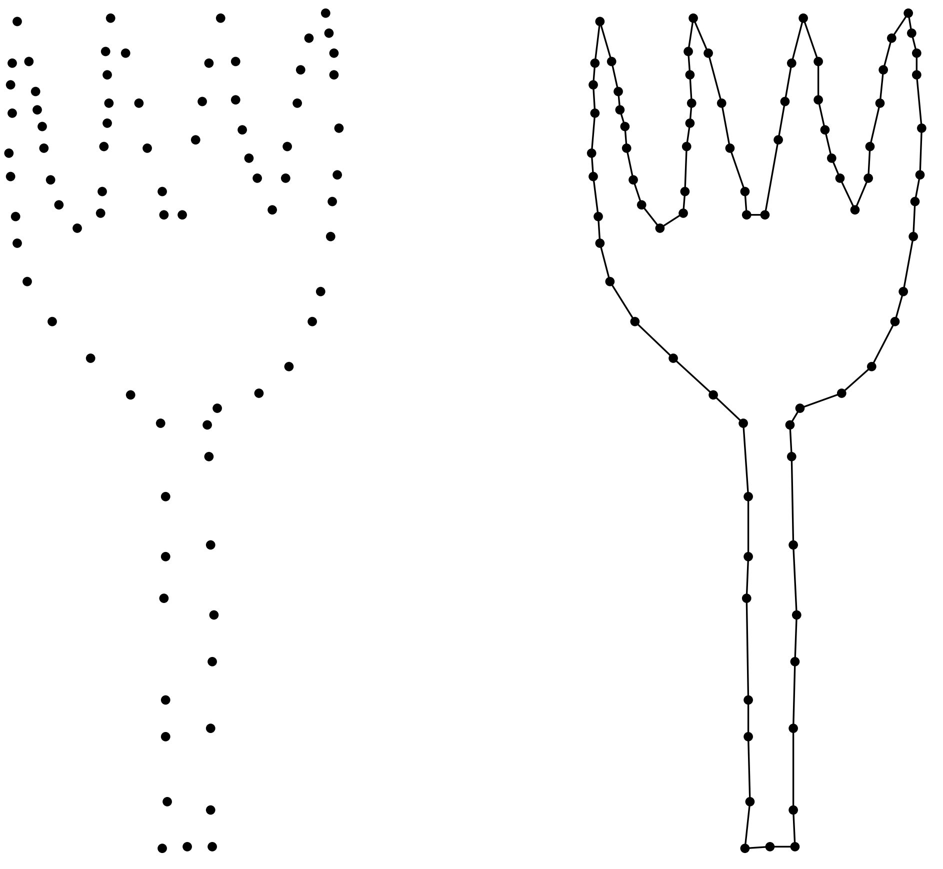
Curve Reconstruction: An Example



probably, you see more than a set of points

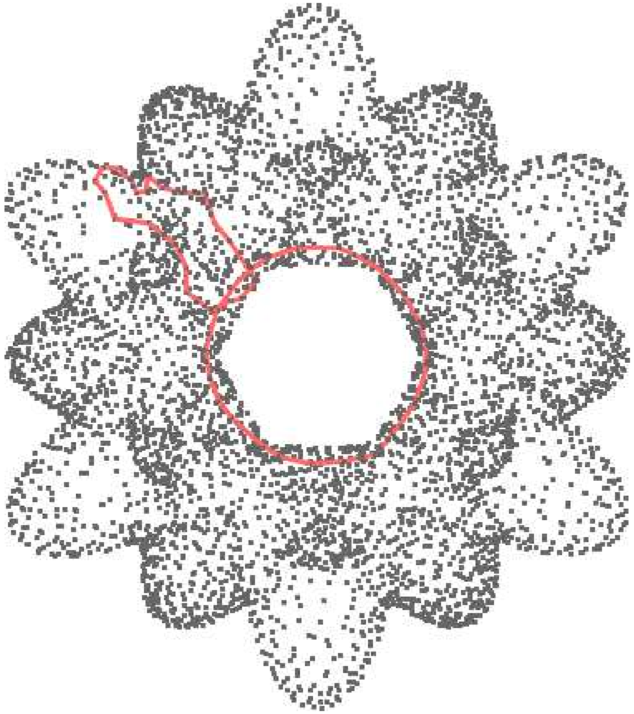
Curve Reconstruction: An Example



reconstructed by algorithm described in

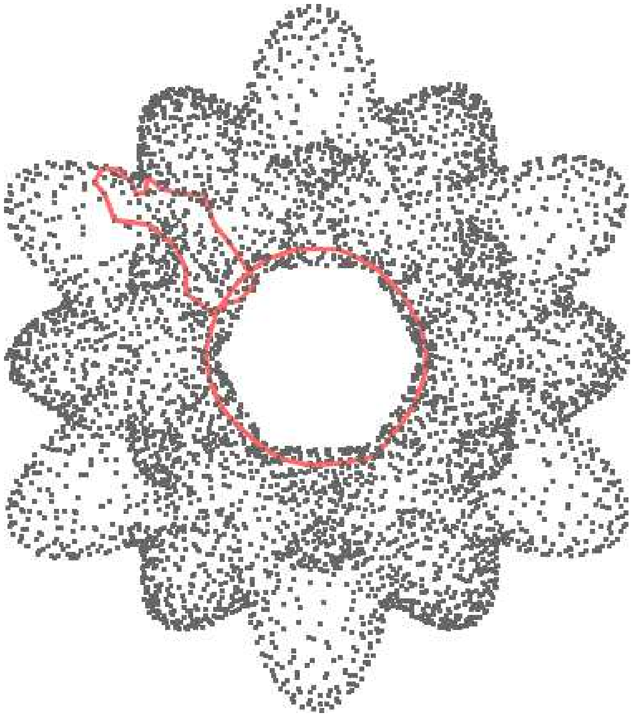
Ernst Althaus and Kurt Mehlhorn: Traveling Salesman-Based Curve Reconstruction in Polynomial Time, SIAM Journal on Computing, 31, pages 27–66, 2002

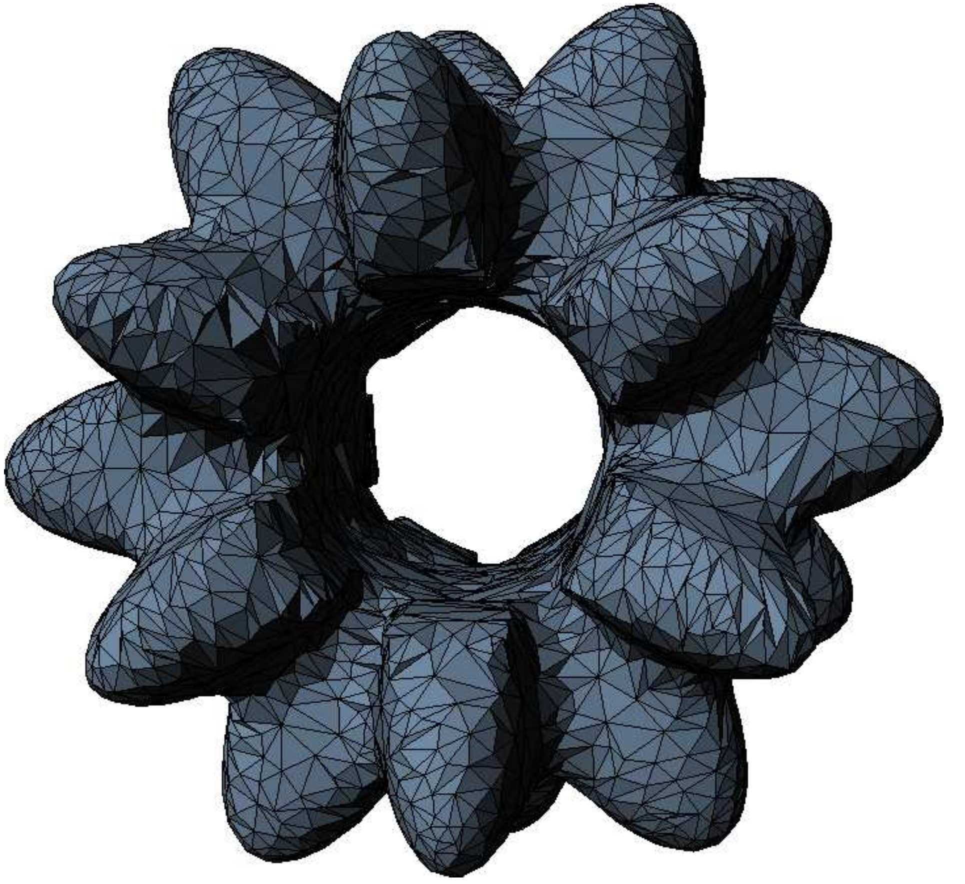
Surface Reconstruction: An Example



probably, you see more than a point cloud

Surface Reconstruction: An Example





reconstructed by algorithm described in

G. Tewari, C. Gotsman, and S. Gortler, Meshing Genus-1 Point Clouds using Discrete One-Forms, Harvard

Technical Report 2005

Problem Definition: Curve and Surface Reconstruction

Input: A finite sample P from an unknown curve $\gamma \in \mathbb{R}^2$ or surface $S \in \mathbb{R}^3$

Output (Curve): $G(P,E)$ where $xy \in E$ iff x and y are adjacent on γ.

Output (Surface): A triangulation of P topologically equivalent and geometrically close to S

The Goal: Algorithms with guarantees (preferably, simple algorithms):

- reconstuction succeeds if
  - $\gamma \in \Gamma$ (= a class of curves) or $S \in \Gamma$ (= a class of surfaces) and
  - P satisfies a certain *sampling condition*.
- low asymptotic (as function of $n = |P|$) and practical complexity

Motivation:

- line drawings from raster images
- surface reconstruction

State of the Art: Curve Reconstruction

- till 97, uniformly sampled smooth closed curves
- 97, non-uniformly sampled smooth closed curves, Amenta/Bern/Epstein, Dey/Kumar
- 99, non-uniformly sampled smooth open and closed curves, Dey/Mehlhorn/Ramos
- 99, TSP reconstructs uniformly sampled non-smooth curves, Giesen
- 00, TSP reconstructs non-uniformly sampled non-smooth curves in polynomial time, Althaus/Mehlhorn
- 01, near-linear time reconstruction of non-uniformly sampled non-smooth curves, Funke/Ramos

smooth curve: tangent everywhere

uniform sample: at least one sample from every curve segment of length ε.

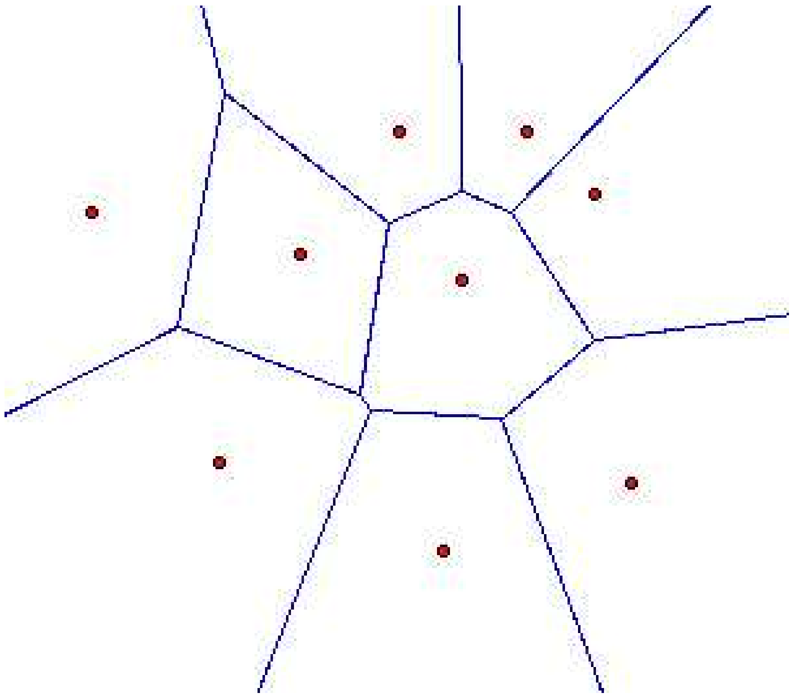
non-uniform sample: sampling rate depends on local features of the curve.

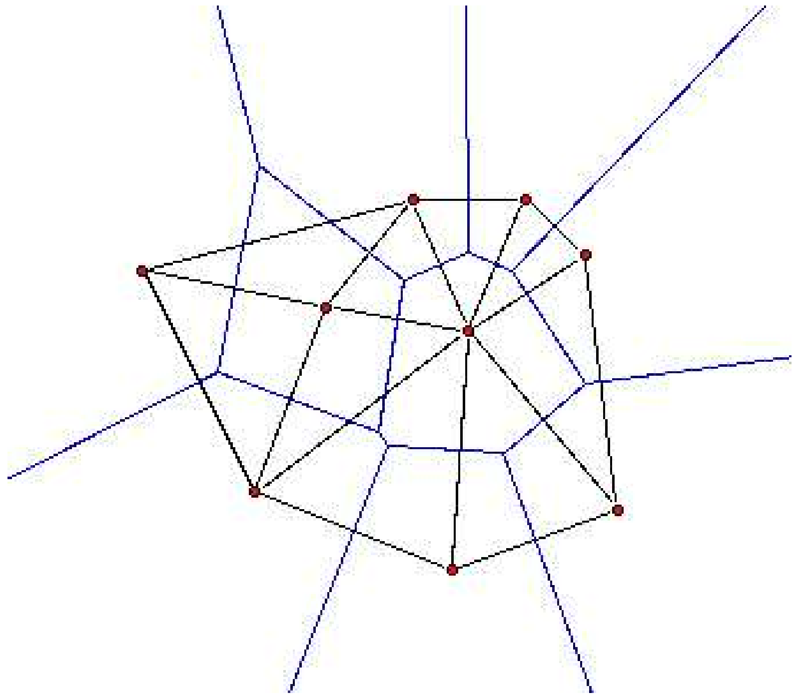
Surface Reconstruction

- till 97, only heuristics
- 98, non-uniformly sampled smooth closed surfaces, Amenta/Bern, Boissonnat/Casals
- 00, non-uniformly sampled smooth closed surfaces, topological and geometric guarantee, Amenta/Choi/Dey/Leekha
- 02, near-linear time, non-uniformly sampled smooth closed surfaces Funke/Ramos
- 02 – , various attempts to generalize to non-smooth surfaces and surfaces with boundary many

The Cocone Algorithm I

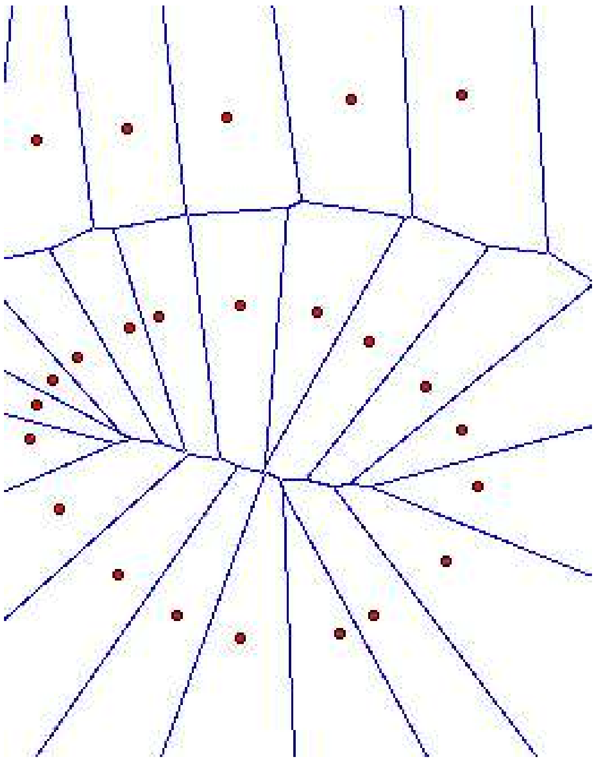
- Amenta/Choi/Dey/Leekha
- Voronoi and Delaunay Diagram of P
- Voronoi region $V(p)$ of $p \in P$ consists of all points in the plane which are closer to p than to any other point in P.
- Delaunay diagram of P: connect two points in P if their Voronoi regions share an edge





The Cocone Algorithm II

- Voronoi region $V(p)$ of $p \in P$ consists of all points in the plane which are closer to p than to any other point in P.
- Pole of $p \in P$: direction to vertex of $V(p)$ farthest from p
- Cocone idea: pole is a good estimate of curve normal
- select edges of Delaunay diagram which are (almost) perpendicular to pole and then do a bit more



run demo

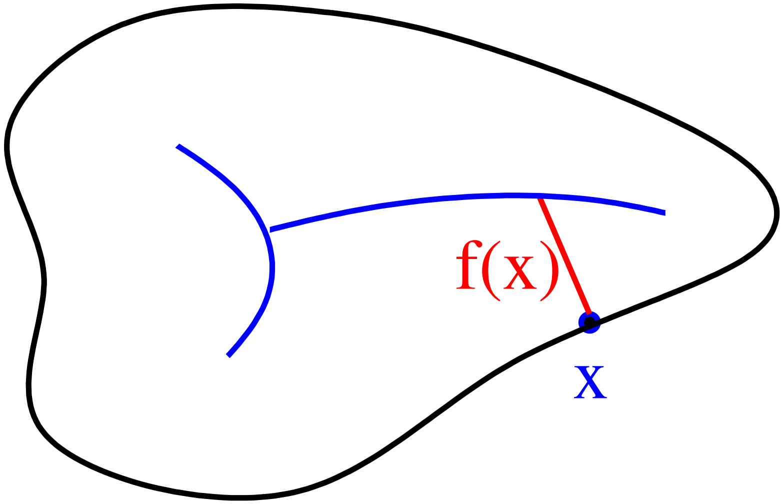
The Cocone Algorithm III

- generalizes nicely to three dimensions
- algorithm reconstructs a triangulation which is topologically equivalent and geometrically close to S if the following sampling condition holds:

for every $x \in S$ there is a $p \in P$ with

$$dist(x, p) \leq 0.1 f(x)$$

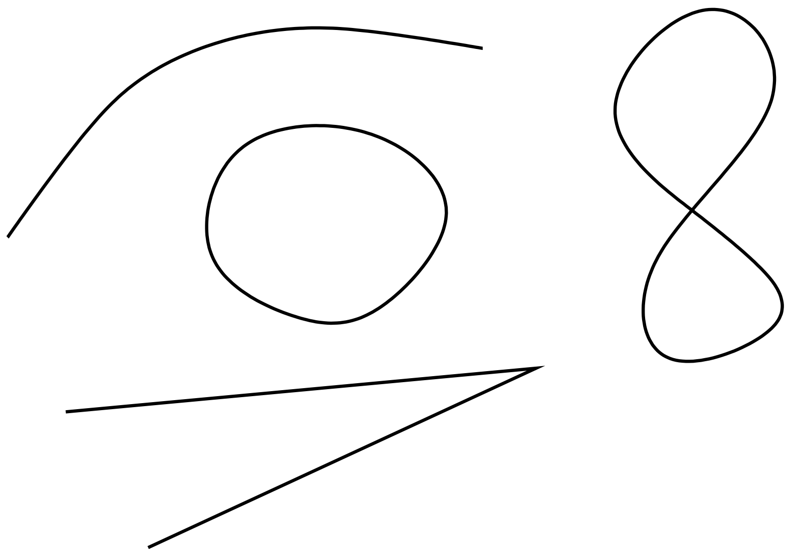
where $f(x)$ is the distance of x to the medial axis of S.



- running time is quadratic $O(n^2)$, Funke/Ramos improved to $O(n \log n)$
- algorithms work for large samples, n up to 10^6

BUT

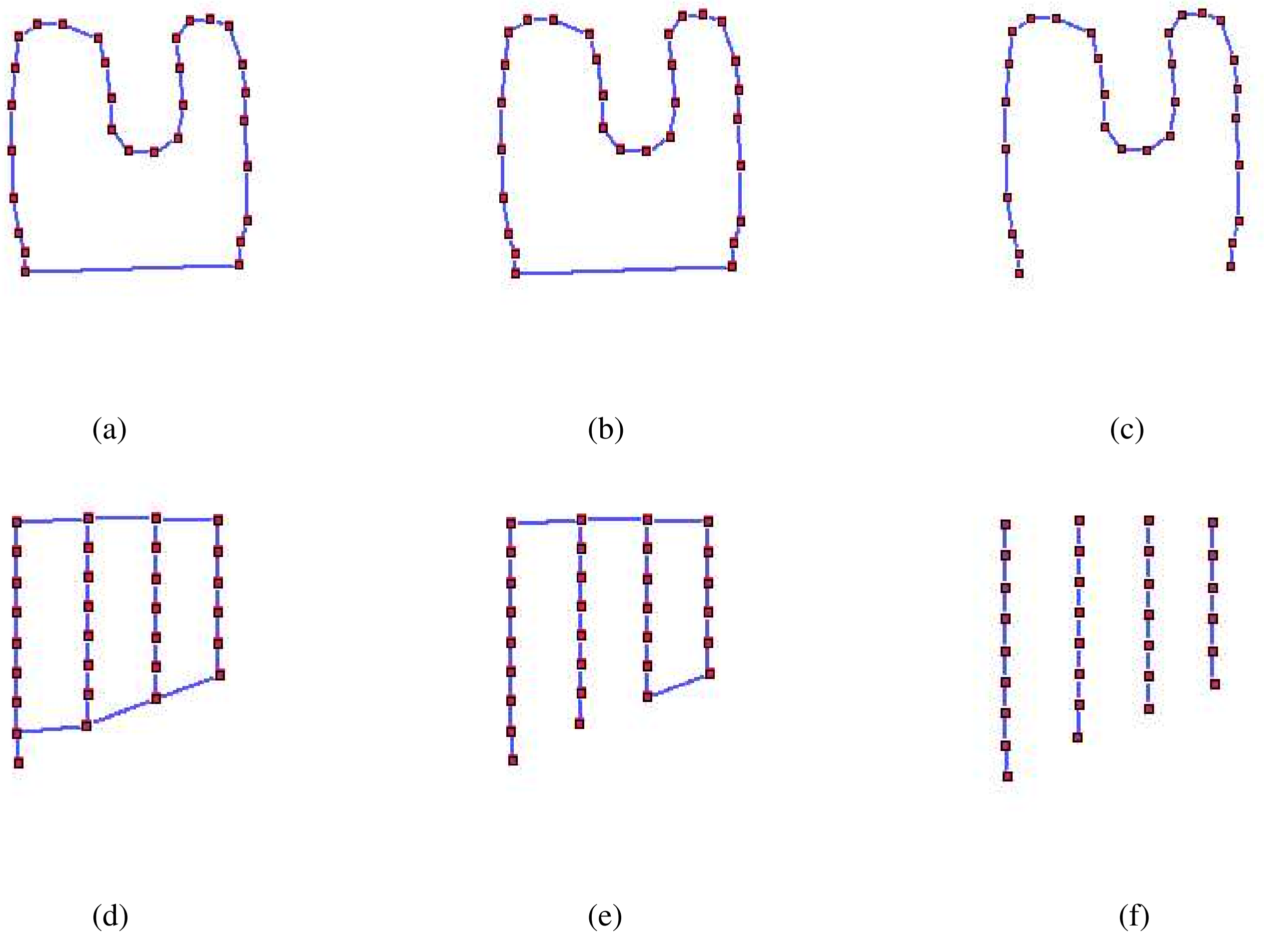
- open and closed curves
- sharp corners
- branching points



- branching points are open, open and closed curves and non-smooth curves can be handled

Open and Closed Curves

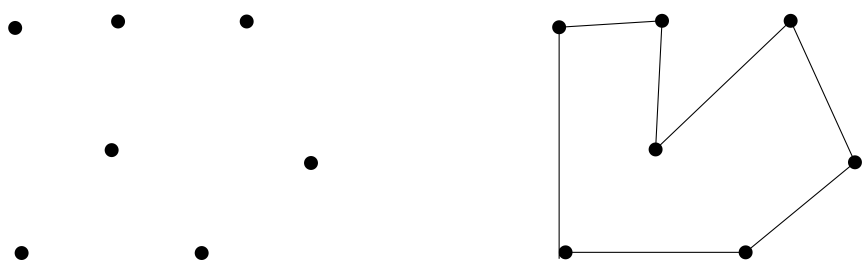
DMR (Compgeo 99): A variant reconstructs non-uniformly sampled open and closed curves.



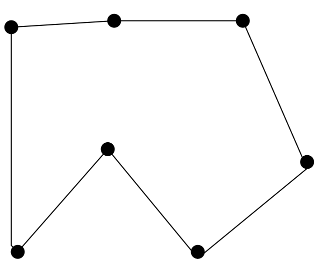
DK ABE DMR

The Traveling Salesman Problem

- Problem Definition (Geometric TSP)
  - given a set P of points in the plane
  - find the shortest closed tour passing through all the points
- Graphic TSP
  - given a graph $G = (V, E)$ with integral edge weights
  - find a shortest closed tour passing through all the vertices
- computationally very difficult
- no algorithm with polynomial running time is known
- smallest unsolved problems have only a few hundred points or nodes
- graphic TSP is NP-complete, geometric TSP is NP-hard

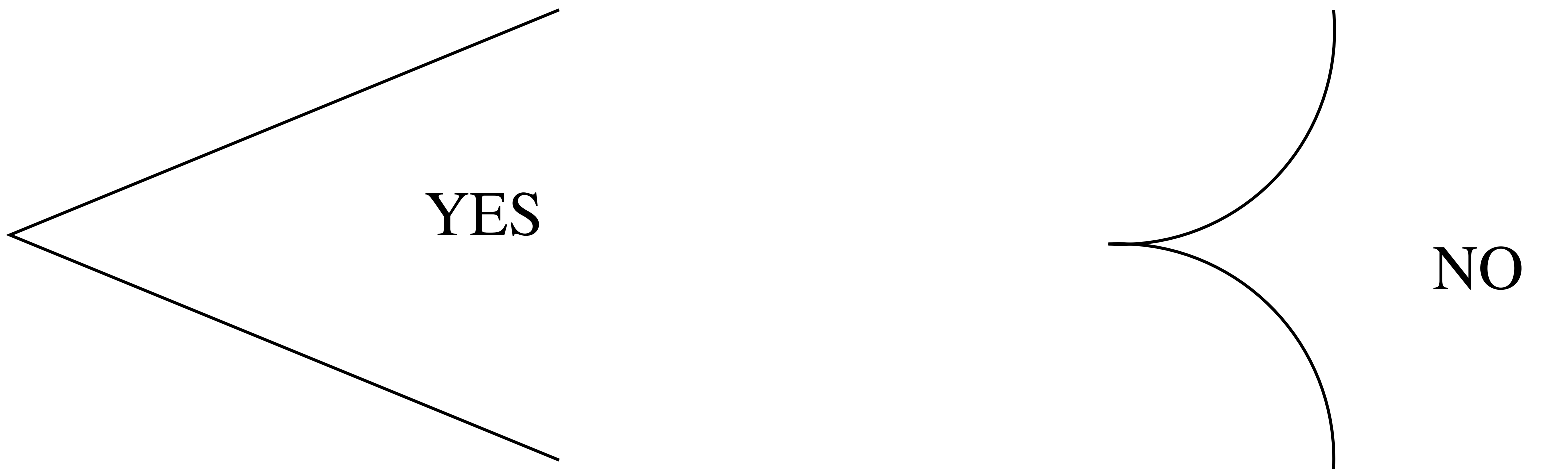


or



Sharp Corners

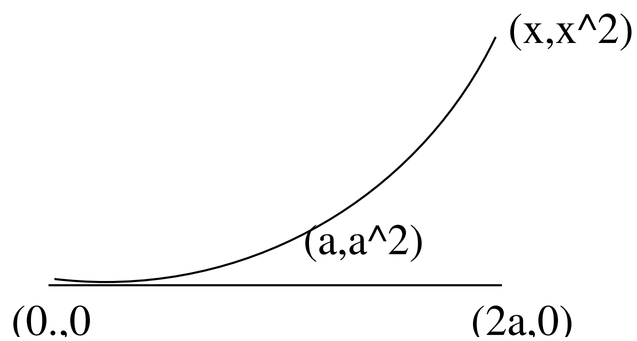
semi-regular curve: left and right tangent exists and turning angle $< \pi$.



Giesen (Compgeo99): TSP reconstructs uniformly-sampled semi-regular curves, i.e.,
for every semi-regular curve γ there is an $\varepsilon > 0$:
if P contains at least one point from every curve segment of length ε then $TSP(P)$ reconstructs γ.

- exact TSP is required, approximate TSP will not do
- result is structural, not algorithmic

TSP does not work for turning angle equal to zero



- O = origin, x-axis, parabola $y = x^2$
- let x be such that $dist(O,(x,x^2)) = dist(O,(2a,0))$
- order on curve = $(2a,0) - (0,0) - (a,a^2) - (x,x^2)$

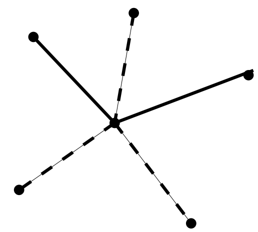
- wrong order = $(2a,0) - (a,a^2) - (0,0) - (x,x^2)$

wrong order gives shorter length than correct order for arbitrarily small a since (a,a^2) lies on the wrong side of the angular bisector

An Integer Linear Program for TSP

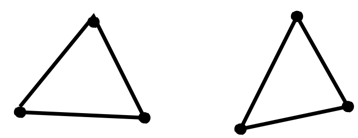
x_e variable for edge $e = uv$ $\qquad$ $c_e =$ Euclidean length of $e = uv$

minimize $\quad \sum_e c_e x_e \quad$ subject to



$$\sum_u x_{uv} = 2 \quad \text{for all } v \in P$$

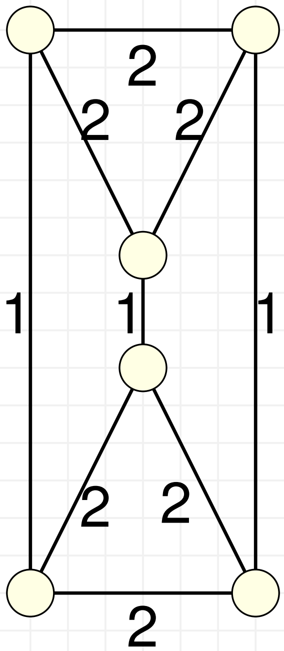
$$\sum_{\{uv;\ u \in R, v \notin R\}} x_{uv} \geq 2 \quad \text{for all } R \subset P \text{ with } \emptyset \neq R \neq P$$

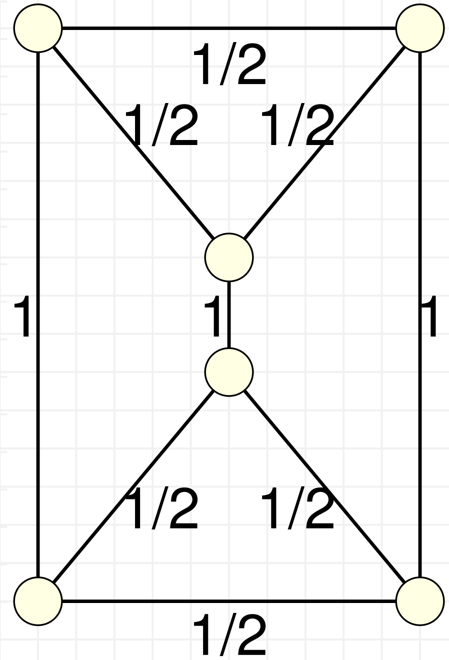


$$0 \leq x_e \leq 1, \text{ integral}$$

- subtour LP, remove integrality constraint
- subtour LP can be solved in polynomial time
- optimal solution of subtour LP is (in general) fractional

Fractional Optimal Solution

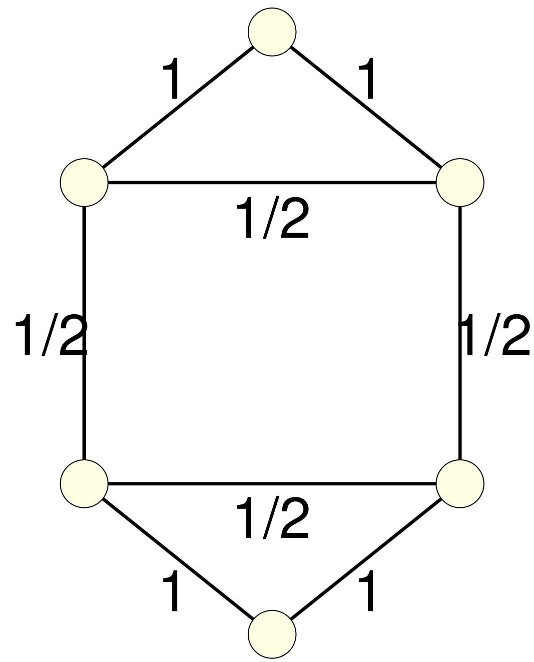




- left side: edges weights right side: optimal solution to LP
- optimal tour has cost $4 \cdot 2 + 2 \cdot 1 = 10$.
- fractional solution has cost $6 \cdot 0.5 \cdot 2 + 3 \cdot 1 \cdot 1 = 9$.

A Cutting Plane Algorithm

- Solving the LP exponentially many constraints
  - solve LP without subtour elimination constraints
  - check for violated subtour elimination constraint
    - let x_e be the solution of the LP
    - compute minimum cut in (P, E, x_e)
    - a cut of value < 2 yields a violated constraint
    - add and repeat



  - runs in polynomial time with Ellipsoid method
  - is practically efficient with simplex method
- Solving the ILP
  - When LP has fractional solution, branch on fractional variable

The Main Result in Althaus/Mehlhorn

Experimental observation: optimal solution of subtour LP is integral

Theorem(AM): Let γ be a semi-regular curve. If P is a sufficiently dense sample of γ then

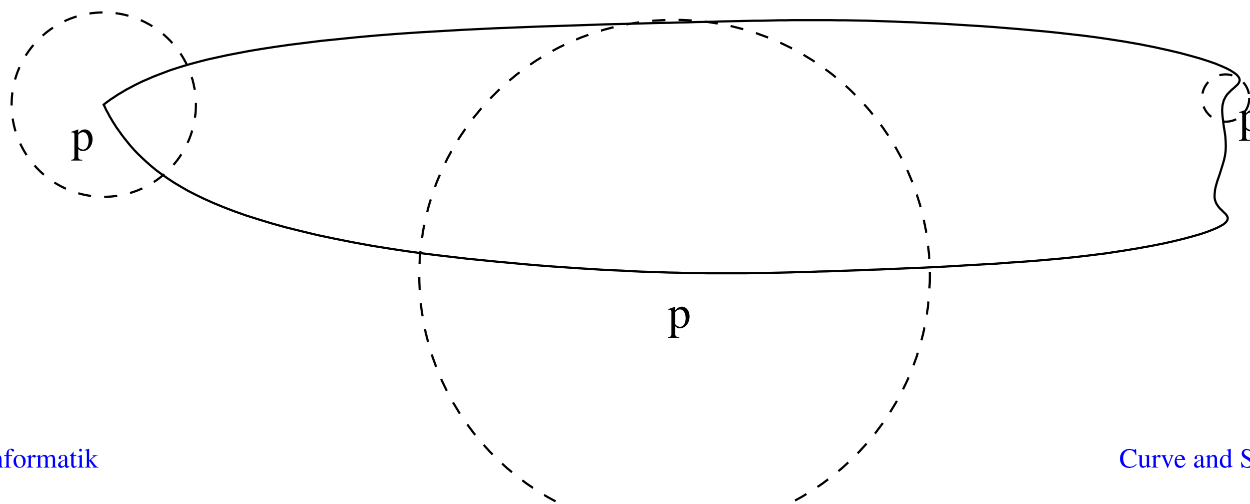
- optimal solution of subtour LP is integral (and hence a tour)
- can be found in polynomial time

Proof Idea

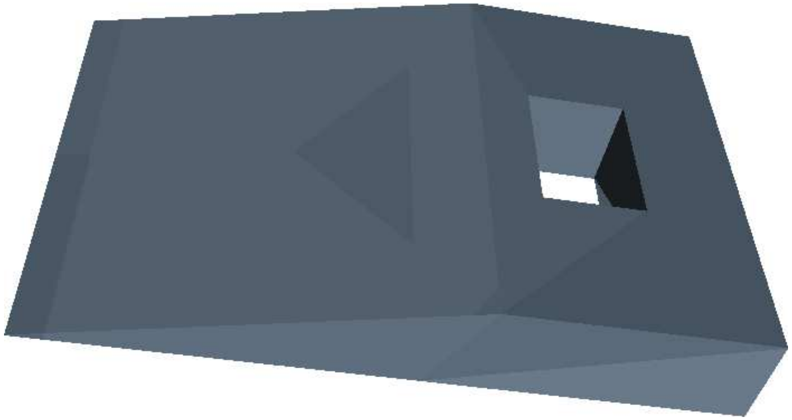
- exploit duality of subtour LP and Held-Karp bound
- show that Held-Karp bound yields a tour.

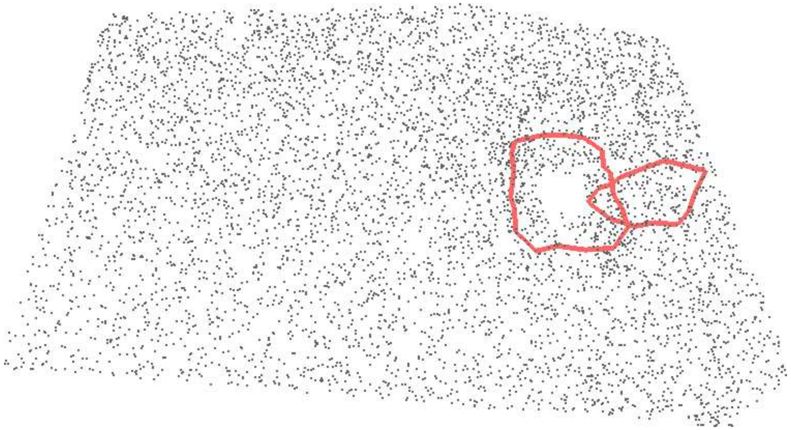
How Dense is Sufficiently Dense?

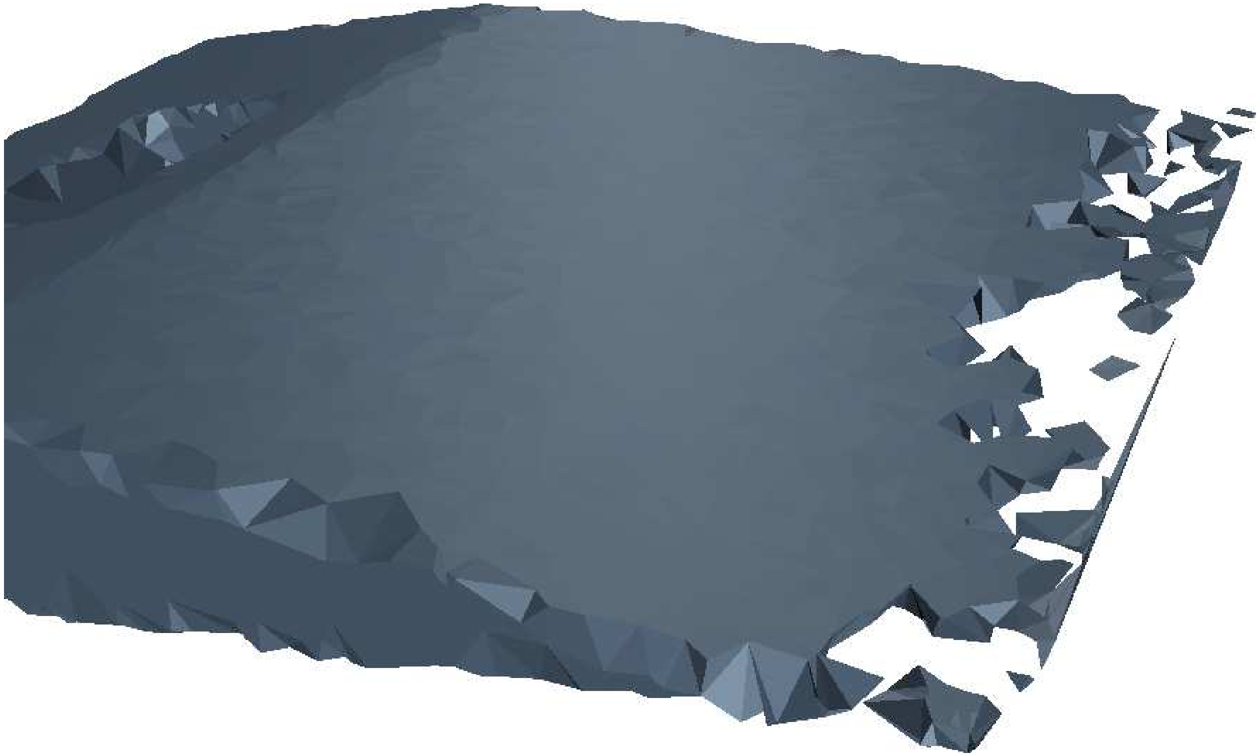
1. for every corner (= discontinuity) p, let $R(p)$ be largest such that
   (a) legs of corner turn by at most 10° within $B(p, R(p))$
   (b) curve is connected inside the disk
2. must have at least one sample point on each leg within $B(p, R(p)/4))$
3. break curve into smooth pieces by removing the disks $B(p, R(p)/8))$
4. for every p in one of the smooth parts, let $R(p)$ be largest such that
   (a) curve turns by at most 60°) within $B(p, R(p))$
   (b) curve is connected inside the disk
5. must have at least one sample point within $B(p, R(p)/4))$



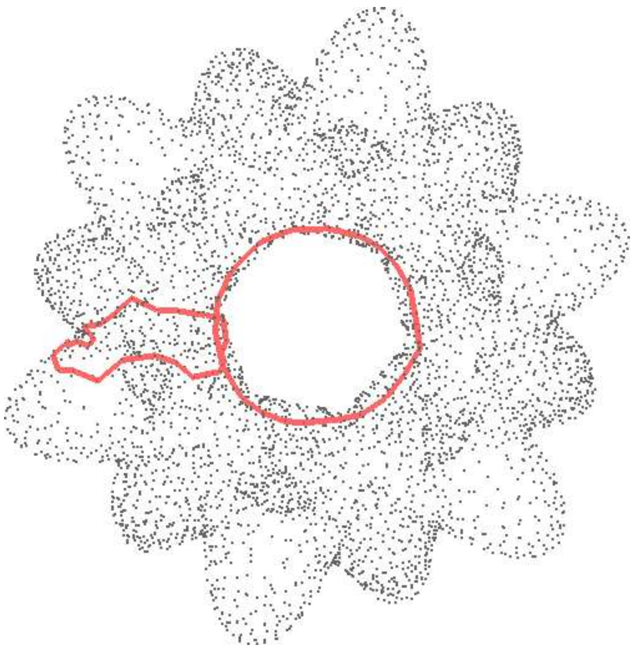
Beyond Smooth Surfaces: Cocone Reconstruction

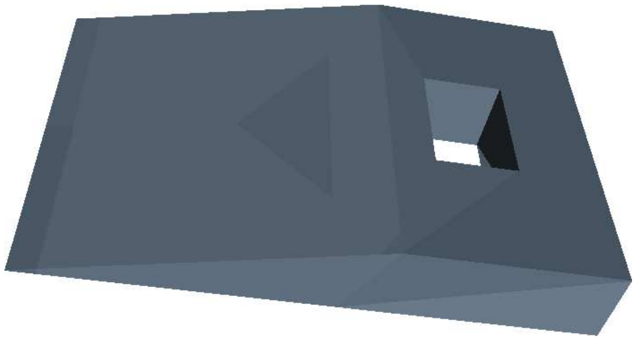


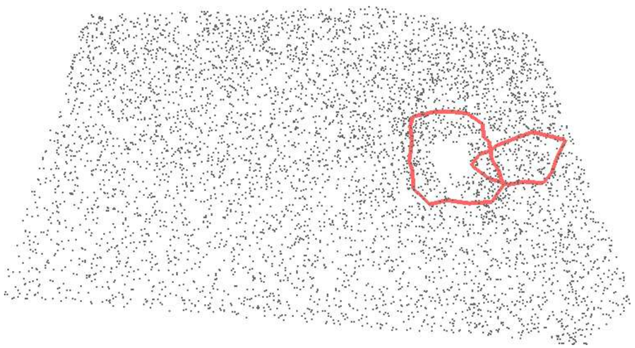




Beyond Smooth Surfaces: Genus Detection I







- genus g of a closed surface = sphere + g handles
- examples are genus one surfaces, i.e., homeomorphic to a torus
- genus detection: compute g and $2g$ cycles spanning the non-trivial cycles

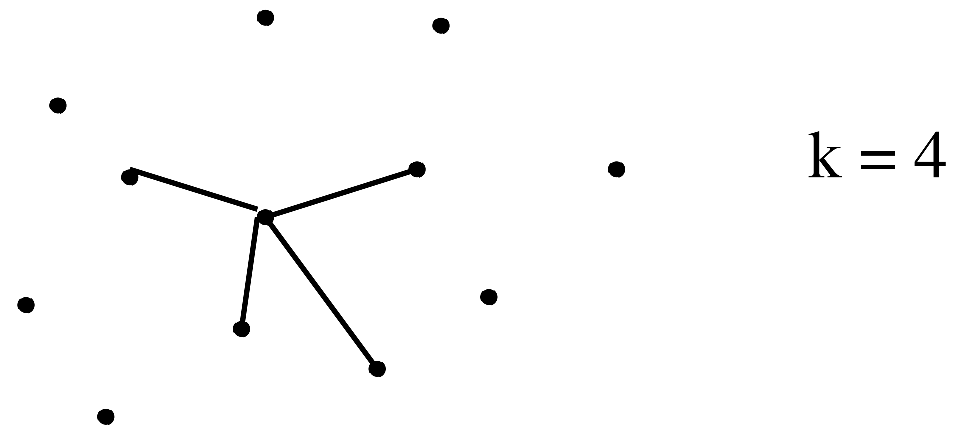
Minimum Cycle Basis (MCB) in Graphs

- (generalized) cycle in a graph: a set of edges with respect to which every node has even degree
- addition of two cycles = symmetric difference

- cycles form a vector space (over field of two elements) under addition
- minimal cycle basis (MCB) = a set of cycles spanning all cycles and having minimal total length
- can be computed efficiently (Kavitha/Mehlhorn/Michail/Paluch, SODA 04, $O(m^2n + mn^2 \log n)$)

MCBs in Nearest Neighbor Graph

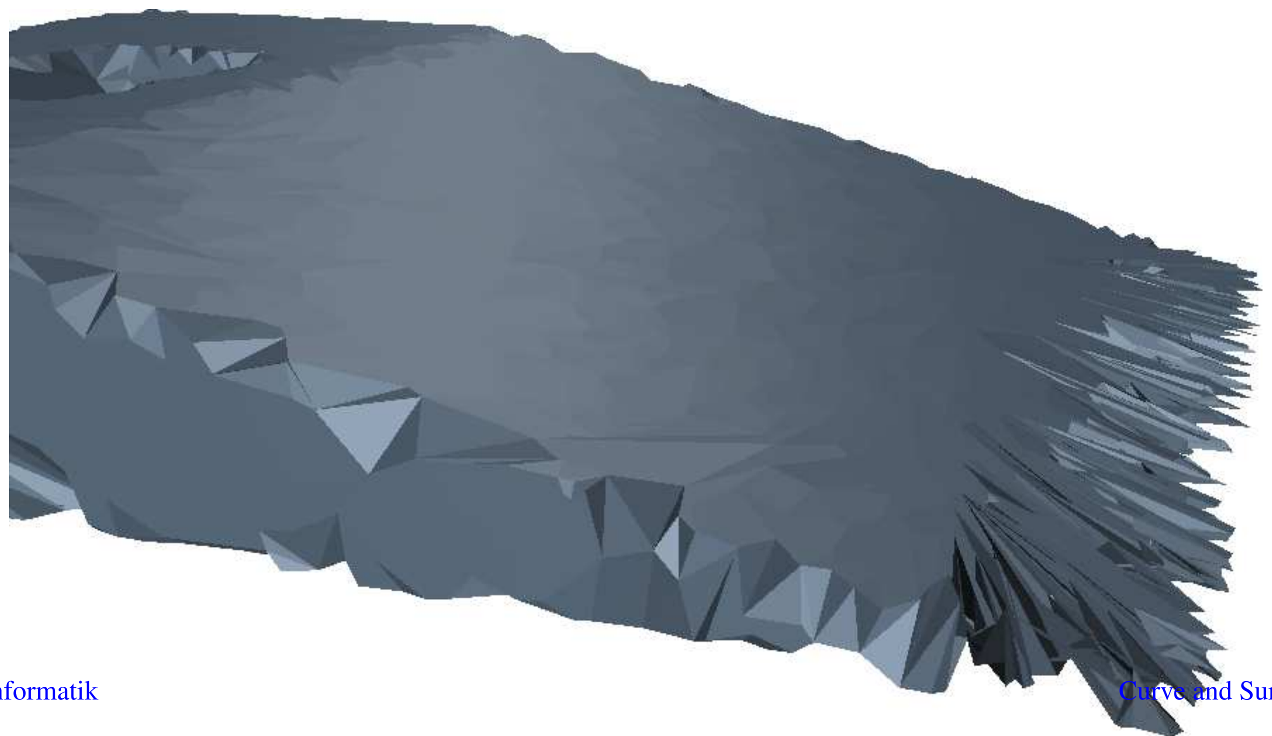
- Nearest Neighbor Graph G_k on P (k integer parameter)
  - connect u and v is v is one the k points closest to u and vice versa



  - easy to construct
- Theorem (Gotsman/Kaligossi/Mehlhorn/Michail/Pyrga 05): if S is smooth, P is sufficiently dense, and k appropriately chosen: MCB of $G_k(P)$ consists of short (lenght at most $2k+3$) and long (length at least $4k+6$) cycles. Moreover, the short cycles span the space of trivial cycles and the long cycles form a homology basis.
- alg also provably works for some non-smooth surfaces (theorem to be formulated)

Beyond Smooth Surfaces: Reconstruction

- Tewari/Gotsman/Gortler have an algorithm to reconstruct genus-1 surfaces if a basis for the trivial cycles of $G_k(P)$ is known.
- algorithm constructs a genus-1 triangulation of P
- no geometric guarantee
- our algorithm computes a basis for the trivial cycles of $G_k(P)$
- together the algorithms reconstruct genus-1 surfaces



Summary

- curves
  - efficient algs are known for open and closed, smooth and non-smooth curves
  - branching points are open problem
  - noise is partially solved
- surfaces
  - efficient algs are known for closed smooth surfaces
  - noise is partially solved
  - open
    - surfaces with boundary
    - non-smooth surfaces

Thank you for your attention