Who Gets What?

Fair Division of Indivisible Goods

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November 2022

An Everyday Problem

Ad of a Saarbrücken law firm:
Your brother enjoys your bequest in the sun.

The websites Spliddit and Fair Outcome offer algorithms for fair division problems, e.g.,

- dividing goods (divorce settlement, bequest),
- splitting rent,
- dividing chores (papers to review or household duties).

Two Agents: Cut-and-Choose

Divide and choose is mentioned in the Bible, in the Book of Genesis (chapter 13). When Abraham and Lot come to the land of Canaan, Abraham suggests that they divide it among them. Then Abraham, coming from the south, divides the land to a "left" (western) part and a "right" (eastern) part, and lets Lot choose. Lot chooses the eastern part which contains Sodom and Gomorrah, and Abraham is left with the western part which contains Beer Sheva, Hebron, Beit El, and Shechem.

The United Nations Convention on the Law of the Sea applies a procedure similar to divide-and-choose for allocating areas in the ocean among countries. A developed state applying for a permit to mine minerals from the ocean must prepare two areas of approximately similar value, let the UN authority choose one of them for reservation to developing states, and get the other area for mining.

Two Agents: Cut-and-Choose

Agents 1 partitions the set of goods into two bundles. Agent 2 chooses the bundle, they like more.

Agent 1 is happy, because they have done the splitting in their best interest, i.e, maximized the value of the less valuable bundle. Agent 2 is happy, because they will choose their preferred bundle.

Agent 1 is unhappy, because they had to do the splitting. He is only guaranteed the smaller part of the best partition. Agent 2 is unhappy, because he can only choose among the parts decided upon by agent 1. . . . .

Maybe better:

- Both agents split.
- A coin toss decides who plays the role of agent 1.

Ape story.

Discrete Fair Division

Given

- Set $[n]$ of n agents, $i, j \dots$ are agents.
- Set M of m indivisible goods. house, car, toothbrush, ...
- Valuation of agent i: $v_i : 2^M \to \mathbb{R}_{\geq 0}$.

  $v_i(S)$ = value of bundle S to agent i,
  $v_i(\emptyset) = 0$ and $v_i(A) \leq v_i(B)$ if $A \subseteq B$.
- Valuation is additive if $v_i(S) = \sum_{g \in S} v_i(g)$ for all bundles S.
- Valuations are additive, if not said otherwise.

Find: A fair partition $X = \langle X_1, X_2, \dots, X_n \rangle$ of M.
Maybe better: A partition satisfying certain fairness criteria.

X_i is the bundle assigned to agent i.

Fair division of chores is equally interesting.

Three Notions of Fairness

- Envy-based notions: EF (envy-free), EF1, EFX, ...
- Share-based notions: Proportionality, MaxiMin-Share, ...
- Nash Social Welfare: assigns a fairness-score to each allocation

Own work in EC20, EC21, JACM 22, FSTTCS 20, SODA 21, AAAI 22, SICOMP 22, arXive 22, together with H. Akrami, B. Chaudhury J. Garg, M. Hoefer, T. Kavitha, R. Mehta, P. Misra, M. Schmalhofer, A. Sgouritsa, G. Shahkarami, Q. Vermande, and E. van Wijland

Many more papers by former members of D1: Bhaskar Chaudhury, Jugal Garg, Martin Hoefer, Yun Kuen Cheung, and Naveen Garg.

Envy-Based Notions

Agent i considers an allocation fair, if nobody gets more than they do, i.e., $v_i(X_j) \leq v_i(X_i)$ for all i and j.

Envy-freeness is too much to ask: One good, two agents that both like the good.

Relaxations:

Envy-Based Notions

Agent i considers an allocation fair, if nobody gets significantly more than they do, i.e., $v_i(X_j) \leq v_i(X_i)$ for all i and j.

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Relaxations:

- EF1 (envy-freeness up to one good): envy goes away after the removal of the most valuable good, i.e., for all i and j, there is $e \in X_j$ such that $v_i(X_j - e) \leq v_i(X_i)$.
- EF1 allocation always exists (Lipton et al, 2004).
- EF1 is unsatisfactory.

$\{$ car $\}$ $\{$ house, toothbrush $\}$

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Relaxations:

- EFX (envy-freeness up to any good) envy goes away after the removal of any good, i.e.,
  for all i and j and all $e \in X_j$, we have $v_i(X_j - e) \le v_i(X_i)$.
- EFX is a demanding notion: envy must go away after the removal of the least valuable good.

  $$\{\text{car}\} \qquad \{\text{house, toothbrush}\}$$

  is not EFX but

  $$\{\text{car, toothbrush}\} \quad \{\text{house}\}$$

  is.

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  for all i and j and all $e \in X_j$, we have $v_i(X_j - e) \leq v_i(X_i)$.
- EFX exists for 3 agents and additive valuations (Chaudhury-Garg-M, 2019), 3 agents and two general valuations (Akrami et al, 2022).
- Open Problems: EFX for 4 agents, n agents, EFX for 3 agents and general valuations, tEFX, EFX + share guarantee.

Share-Based Notions

Agents consider an allocation fair, if they get their fair share.

Proportional Share: $v_i(X_i) \geq v_i(\text{all goods})/n$ for all i.

Proportional share is too much to ask.

MaxiMin Share: i partitions and gets the least valuable bundle, i.e., $MMS_i = \max_{(X_1,\ldots,X_n)} \min_\ell v_i(X_\ell)$. Definition involves only v_i.

X is MMS-Partition if $v_i(X_i) \geq MMS_i$ for *all* i.

- Computation of MMS_i is NP-complete.
- Cut and Choose for 2 agents and additive valuations guarantees MMS_1 and proportionality of agent 2.
- MMS-partition is not guaranteed to exist.
- Approximations, i.e, guarantee $v_i(X_i) \geq \alpha MMS_i$ for all i.
  - cannot guarantee $\alpha \geq 1 - \frac{1}{n^4}$.
  - $\alpha = 3/4 + 1/(12n)$, Ghodshi et al. 2019, Garg/Tarek 2021, Hana simplified proof.
  - Open problem: Close the gap.

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Nash Social Welfare (NSW)

Partition the goods into n bundles $X_1, X_2, \ldots, X_n$ so as to maximize

$$NSW(X) := \left(\prod_i v_i(X_i)\right)^{1/n} \qquad \text{geometric mean.}$$

Nash-value $NSW(X)$ ranks allocations and is the only measure that satisfies some natural axioms, e.g., more equal is better.

NSW-optimal allocation is

- invariant under scaling valuations,
- EF1 (Caragiannis/Kurokawa/Moulin/Procaccia/Shah/Wang), and
- EFX for two-valued valuations, i.e., $v_i(g) \in \{1, s\}$ for all i and g.

The Border between P and NP for NSW

- Finding NSW-optimal allocation is NP-complete in general
  - even for two agents and identical evaluations (reduction from subset sum).
  - for three-valued valuations $v_i(g) \in \{0, 1, s\}$ (reduction from 3d-matching).
  - Even hard to approximate.

- What can be done efficiently?
  - 1.45-approximation algorithm (Barman/Rohit/Vaish)
  - zero-one valued is in P; $v_i(g) \in \{0, 1\}$

A dichotomy for two-valued instances: $v_i(g) \in \{1, s\}$ for all i and g, $s \in \mathbb{Q}$, $s \geq 1$ (Akrami, Chaudhury, Hoefer, M, Schmalhofer, Shahkarami, Vermande, van Wijland, 21 and 22):

- in P if s is a multiple of $1/2$.
- NP-complete if s is not a multiple of $1/2$.

EFX with Charity Caragiannis/Gravin/Huang '19, Chaudhury/Kavitha/M/Sgouritsa SODA '20, Chaudhury/Garg/M/Misra EC '21, Akrami/Alon/Chaudhury/M unpublished

There exists a partition $\langle X_1, X_2, \ldots, X_n, P\rangle$ of the items such that,

- $\langle X_1, X_2, \ldots, X_n\rangle$ is EFX,
- For every i, we have $v_i(P) \leq v_i(X_i)$ and
- $|P| < n$.

and also with sublinear charity: $n^{5/6}$, $n^{2/3}$, $n^{1/2}$.

The improvements are via a connection to extremal graph theory: Consider a graph G with the following property:

- Node set partitions into k sets S_1 to S_k of size d each.
- For all i and all $v \in S_i$: v has an incoming edge from every S_j, $j \neq i$.

Show: if $k \geq f(d)$ then G contains a cycle hitting each S_i at most once. $f(d) = O(d^3), O(d^2), O(d \log d)$.

Best of Both Worlds Guarantees

Find a distribution over a small number of integral allocations with good ex-ante and ex-post guarantees.

Let $X^1, \ldots X^k$ be the integral allocations with $X^\ell = (X_1^\ell, \ldots, X_n^\ell)$ and let p_1 to p_k their probabilities.

- Ex-ante, proportional: $\sum_\ell p_\ell v_i(X_i^\ell) \geq v_i(M)/n$,
- Ex-post:
  - EF1: each X^ℓ is EF1.
  - approximate MaxiMinShare: $v_i(X_i^\ell) \geq \frac{1}{2} MMS_i$ for all i.

Open problem: Stronger ex-post gurantees.