On the isomorphism of two algorithms:
Hu/Tucker and Garsia/Wachs

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In 1969 T.C. Hu and K.C. Tucker succeeded in proving the correctness of Hu's algorithm for constructing optimum binary search trees (cf. Knuth, Vol. 3, p. 447 for a historical account). The proof is extremely complicated and until now no simpler proof was found. T.C. Hu published a simplified proof in 1973; however a key lemma in that paper is wrong (see Tsagarakis).

In 1977 Garsia and Wachs published a new algorithm for constructing optimum binary search trees and gave a fairly short correctness proof for it.

Both algorithms use sequences of real numbers as their underlying data structure. We show that the two algorithms manipulate these sequences in an isomorphic way, i.e. the two algorithms, though completely different and hardly related in wording, are the same as much as the manipulation of the underlying data structure is concerned. One immediate consequence of this result is a simple correctness proof for the Hu/Tucker-algorithm.
II. Definitions:

Both algorithms construct the optimal alphabetic tree for a sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of positive reals in two phases. In phase I a non-alphabetic tree $T$ is constructed with the following properties:

1) the weighted path length of tree $T$ is equal to the weighted path length of an optimal alphabetic tree.

2) let $l_i$ be the depth of leaf $\alpha_i$ in tree $T$, $1 \leq i \leq n$. Then there is an optimal alphabetic tree having $\alpha_i$ at depth $l_i$.

In phase II an optimal alphabetic tree is constructed using the sequence $l_1, l_2, \ldots, l_n$ of depths of the $\alpha_i$. Phase II is the same in both algorithms; we will only be concerned with phase I of the algorithms. It is time-consuming and difficult to validate. We next describe phase I of both algorithms.

II.1 Garsia/Wachs: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a sequence of positive reals.

Definition:

a) Pair $(\alpha_{i-1}, \alpha_i)$ is right-minimal if $\alpha_{i-2} + \alpha_{i-1} - \alpha_{i-1} + \alpha_i < \alpha_{j-1} + \alpha_j$ for all $j > i$.

b) Pair $(\alpha_{i-1}, \alpha_i)$ is right-minimum if $\alpha_{k-1} + \alpha_k - \alpha_{i-1} + \alpha_i$ for all $k < i$ and $\alpha_{i-1} + \alpha_i < \alpha_{j-1} + \alpha_j$ for all $j > i$.

The right-minimum pair is uniquely determined. It is always right-minimal. Phase I of the G.W. algorithm works iteratively: Each iteration works as follows:

Let $(\alpha_{i-1}, \alpha_i)$ be the right-minimum pair and let $k \geq 0$ be such that $\alpha_{i-1} + \alpha_i > \alpha_{i+j}$ for $1 \leq j \leq k$ and $\alpha_{i-1} + \alpha_i \leq \alpha_{i+k+1}$ or $i+k = n$. A new sequence is formed by deleting $\alpha_{i-1}$ and $\alpha_i$ and inserting their sum $\alpha_{i-1} + \alpha_i$ immediately to the left of $\alpha_{i+k+1}$.
(at the right end if \( i+k = n \)), i.e. the new sequence is
\[ a_1 \ a_2 \ \ldots \ \ a_{i-2} \ a_{i+1} \ \ldots \ a_{i+k} \ a_{i-1} + a_j \ a_{i+k+1} \ \ldots \ a_n \]

Figure 1 shows the execution of the G.W.-algorithm on the sequence 5, 3, 3, 2, 2, 2, 3, 4 (taken from [GW]). Untangling the lines in Fig. 1 gives the tree of Fig. 2, i.e. both trees have the same solid lines.

II.2 Hn/Tucker: The H.T.-algorithm also works on sequences of reals \( a_1, a_2, \ldots, a_n \). In addition, each real is in either one of two states: it is either a circle or a square. Such sequences are called H.T.-sequences. In the initial sequence all reals are squares. We draw such sequences by enclosing each real in the appropriate geometric figure.

**Definition:**

a) A pair \((a_i, a_k)\) with \( i < k \) is tentative-connecting (T.C.) if \( a_j \) is a circle for \( i < j < k \).

b) A pair \((a_i, a_j)\) is a minimal T.C. pair, if it is T.C. and \( a_i + a_j \leq a_h + a_k \) for all T.C. pairs \((a_h, a_k)\).

c) A pair \((a_i, a_j)\) is the minimum T.C. pair (min. T.C. pair) if it is a minimal T.C. pair and for all other minimal T.C. pairs \((a_h, a_k)\) either \( h < i \) or \( h = i \) and \( k > j \).

**Example:** In the sequence

\[
\begin{array}{cccccc}
   a_1 & a_2 & a_3 & a_4 & a_5 \\
   3   & 1   & 2   & 2   & 2
\end{array}
\]

the pairs \((a_2, a_3)\) and \((a_2, a_4)\) are minimal TC-pairs, the pair \((a_2, a_3)\) is the minimum TC-pair.
Phase I of the H.T.-algorithm works iteratively. Each iteration works as follows:

Let \((\alpha_i, \alpha_j)\) be the minimum T.C. pair. Form a new sequence by deleting \(\alpha_i\) and replacing \(\alpha_j\) by the circle \(\alpha_i + \alpha_j\).

Figure 3 shows the execution of the H.T.-algorithm on the initial sequence 5, 3, 3, 2, 2, 3, 3, 4. Untangling this tree also gives the tree of Fig. 2., i.e. both algorithms performed the same sequence of combinations in the same order!!! They manipulate the data structure in an isomorphic way.

III. Proof of Equivalence

In this section we want to show that the two algorithms perform the identical sequence of combinations. The two algorithms differ in two respects: Only neighboring reals can be combined by the G.W.-algorithms, whilst tentative connecting reals can be combined by the H.T.-algorithm. The G.W.-algorithm reorders the sequence after a combination and the H.T.-algorithm does not.

We introduce a third algorithm, the M.T.-algorithm, which shows features of both, and show that it is equivalent to the G.W.- and the H.T.-algorithm.

M.T.-algorithm: Let \((\alpha_i, \alpha_j)\) be the minimum T.C. pair and let \(k > 0\) be such that \(\alpha_{j+k+1} > \alpha_i + \alpha_j > \alpha_{j+k}\) for all \(0 < \ell < k\). Form a new sequence by deleting \(\alpha_i\) and \(\alpha_j\) and inserting the circle \(\alpha_i + \alpha_j\) immediately to the left of \(\alpha_{j+k+1}\).

Figure 4 shows execution of the M.T.-algorithm on initial sequence 5,3,3,2,2,3,4. Note that Figure 4 is the same as Figure 1 except for the fact that the reals are enclosed in circles and squares this time. We will first show that the M.T.- and the G.W.-algorithm are equivalent.
Definition:

A H.T.-sequence $\beta_1, \ldots, \beta_n$ (i.e. a sequence of reals enclosed in squares and circles) is an actual M.T.-sequence if there is some initial sequence $\alpha_1, \ldots, \alpha_m$ consisting of squares such that some number of iterations of M.T. produces $\beta_1, \ldots, \beta_n$.

Lemma 1:

a) Let $\alpha_1, \ldots, \alpha_n$ be an actual M.T.-sequence and let $\alpha_i$ be a circle. Then $\alpha_i < \alpha_{i+1}$.

b) Let $\beta_1, \ldots, \beta_n$ be an actual M.T.-sequence. Then the minimum T.C.-pair of that sequence is the right-minimum pair of that sequence.

Proof: We prove a) and b) simultaneously by induction on the number $m$ of iterations. For $m = 0$ a) is trivial since there are no circles. Then b) follows immediately from the definition of minimum T.C.-pair and right-minimum pair. Suppose now the claim is true after $m - 1$ iterations, and let $\alpha_1, \ldots, \alpha_n$ be an actual M.T.-sequence obtained after $m - 1$ iterations. Since b) is true for that sequence the minimum T.C.-pair is $(\alpha_i, \alpha_1)$ for some $i$. Let $k \geq 0$ be such that $\alpha_{i+k+1} \geq \alpha_{i-1} + \alpha_i > \alpha_{i+j}$ for $1 \leq j \leq k$.

Then the sequence after $m$ iterations is

$$\alpha_1, \ldots, \alpha_i-2, \alpha_i+1, \ldots, \alpha_{i+k}, \alpha_{i-1} + \alpha_i, \alpha_{i+k+1}, \ldots, \alpha_n$$

We only have to show: if $\alpha_{i-2}$ is a circle then $\alpha_{i-2} \leq \alpha_{i+1}$.

If $\alpha_{i-2}$ is a circle then $\alpha_{i-2} \leq \alpha_{i-1}$ since a) is true for sequence $\alpha_1, \ldots, \alpha_n$. Since $(\alpha_{i-1}, \alpha_1)$ is right-minimum we have $\alpha_{i-1} + \alpha_i \leq \alpha_i + \alpha_{i+1}$ and hence $\alpha_{i-1} < \alpha_{i+1}$. This shows $\alpha_{i-2} < \alpha_{i+1}$ and proves a).
Let $\beta_1, \beta_2, \ldots, \beta_n$ be any M.T.-sequence with:

if $\beta_i$ is a circle then $\beta_i \leq \beta_{i+1}$. Let $(\beta_h, \beta_j)$ be the minimum T.C.-pair. Then $\beta_k$ is a circle for $h < k < j$ and hence $\beta_k \leq \beta_j$ for $h < k < j$. Hence $\beta_h + \beta_{h+1} \leq \beta_h + \beta_j$ and since $(\beta_h, \beta_j)$ is the minimum T.C.-pair we must have $j = h+1$. This proves b).

\[ \square \]

**Corollary:** Algorithms M.T. and G.W. perform exactly the same sequences of combinations.

**Proof:** By part b) of the lemma above the minimum T.C.-pair of an actual M.T.-sequence is always the right-minimum pair of that sequence. Thus the syntactic difference of the two algorithms disappears in the semantics.

\[ \square \]

It remains to show that algorithms M.T. and H.T. are equivalent. To do so we define an equivalence relation on H.T. sequences and then show that the reordering performed by the M.T. algorithm preserves equivalence.

**Definition:**
Two H.T.-sequences are H.T.-equivalent if phase I of the H.T.-algorithm constructs the same tree from them.

**Example:** $\begin{array}{c}
2 \\
1 \\
1 \\
2
\end{array}$ and $\begin{array}{c}
1 \\
1 \\
2 \\
2
\end{array}$ are H.T.-equivalent since the tree of figure 5 is constructed in both cases.

\[ \text{Fig. 5} \]
We need to show that the reordering performed by the M.T.-algorithm preserves equivalence. Therefore we will compare sequences obtained from the same initial sequence by either \( m \) M.T.-steps or \( m \) H.T.-steps. We show by induction on \( m \) that the two sequences are equivalent. This is trivially true for \( m = 0 \). For the induction step we need one more fact about H.T.-sequences.

Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be any H.T.-sequence and let \( T \) be the tree constructed by algorithm H.T. on input sequence \( \alpha_1, \ldots, \alpha_n \). In our example (Figure 3) interior node \( v \) was constructed before interior node \( w \) if and only if the real in node \( v \) is smaller than the real in node \( w \). A more precise version of this observation appears in Lemma 2. Algorithm H.T. combines in each step the min T.C.-pair, deletes the left constituent of the pair from the actual sequence and replaces the right constituent by the sum. Drawing the execution of algorithm M.T. as in Fig. 3 allows us to introduce the partial ordering "to be to the left of" on the nodes of tree \( T \). In our example (Figure 3) \( \mathbf{6} \) is to the left of \( \mathbf{4} \), \( \mathbf{4} \) is to the left of \( \mathbf{5} \) which in turn is to the left of \( \mathbf{24} \), and so on.

**Lemma 2:** Let \( v \) and \( w \) be interior nodes of tree \( T \) constructed by algorithm H.T. on H.T. sequence \( \alpha_1, \alpha_2, \ldots, \alpha_n \). Let \( v \) be \( \mathbf{a} \) and \( w \) be \( \mathbf{b} \). Then:

\( v \) is constructed before \( w \) iff \( \alpha < \beta \) or \( \alpha = \beta \) and \( v \) is to the right of \( w \).

**Proof:** The clause "\( \alpha < \beta \) or \( \alpha = \beta \) and \( v \) is to the right of \( w \)" defines a linear ordering on the set of nodes of \( T \). Note that equal weighted nodes can be not descendants from one another. Hence it suffices to show: If \( v \) is immediately constructed before \( w \) then \( \alpha < \beta \) or \( \alpha = \beta \) and \( v \) is to the right of \( w \).

Let \( \beta_1, \ldots, \beta_m \) be the H.T. sequence immediately before the construction of \( v \). Let \( \beta_i, \beta_j \) be the min T.C.-pair. Then \( v \) represents \( \alpha := \beta_i + \beta_j \) at the previous position of \( \beta_j \).
Next \( w \) is formed. If \( v \) is a son of \( w \) then we certainly have \( \alpha < \beta \). So suppose \( v \) is not a son of \( w \). Then \( w \) is formed by combining \( \beta_h, \beta_k \) with \( \{h,k\} \cap \{i,j\} = \emptyset \).

**Case 1:** \( \max \{h,k\} < \min\{i,j\} \). Then \( \beta_h, \beta_k \) was T.C. just previous to the construction of \( v \). Hence \( \beta = \beta_h + \beta_k \geq \beta_i + \beta_j = \alpha \). Also \( v \) is to the right of \( w \).

**Case 2:** \( \min\{h,k\} > \max\{i,j\} \). Then \( \beta_h, \beta_k \) was T.C. just previous to the construction. Hence \( \beta = \beta_h + \beta_k > \beta_i + \beta_j = \alpha \) by the definition of \( \min \) T.C. pair.

**Case 3:** \( i < h < j < k \) or \( i < h < k < j \) or \( h < i < j < k \) or \( h < i < k < j \). Then \( \beta_i, \beta_h \) and \( \beta_j, \beta_k \) were T.C. just prior to the construction of \( v \). Hence \( \beta_i + \beta_j > \alpha \) and \( \beta_i + \beta_j > \alpha \). This shows \( \beta_h + \beta_k > \alpha \).

We are now ready to prove the equivalence of algorithms M.T. and H.T..

**Lemma 3:** Let \( S = A \; w \; B \; C \) be an H.T.-sequence with \( A = \beta_1 \beta_2 \cdots \beta_{i-1} \), \( w = \beta_i \), \( B = \beta_{i+1} \cdots \beta_{i+k} \), \( C = \beta_{i+k+1} \cdots \beta_n \) and

1) \( w \) is a circle
2) \( v < w \) for all \( v \) in \( B \)
3) Execution of algorithm H.T. on \( S \) never creates a node of weight exactly \( w \) to the right of \( w \).

Then \( S \) and \( \tilde{S} = A \; B \; w \; C \) are H.T.-equivalent.

**Proof:** Assume otherwise. Let \( S \) be a counterexample of smallest length. We show first that the \( \min \) T.C.-pairs of \( S \) and \( \tilde{S} \) are identical.
a) If \( u, v \not\sim w \) then \( u \) and \( v \) are T.C. in \( S \) if and only if they are T.C. in \( \mathcal{S} \). This follows from the fact that \( w \) is a circle.

b) If \( B \) contains at least two elements then \( w \) is not element of the min T.C.-pair of \( S \) (of \( \mathcal{S} \)). This follows from the fact that all elements of \( B \) have weight strictly smaller than \( w \).

c) If \( B \) contains exactly one element, say \( \beta_{i+1} \), and \( w = \beta_i \) is element of the min T.C. pair of \( S \), then \( \beta_i, \beta_{i+1} \) is the min T.C. pair of \( S \). It is also the min T.C. pair of \( \mathcal{S} \). Suppose \( w = \beta_i \) is part of the min T.C. pair and \( \beta_{i+1} \) is not. Since \( w \) is a circle and \( \beta_{i+1} < \beta_i \) this contradicts the definition of min T.C.-pair. So \( \beta_i, \beta_{i+1} \) is the min T.C. pair of \( S \). It is also T.C. in \( \mathcal{S} \). Conversely, the min. T.C. pair of \( \mathcal{S} \) is T.C. in \( S \). Hence \( \beta_i, \beta_{i+1} \) is also the min T.C. pair of \( \mathcal{S} \).

In any case it follows from a), b) and c) that the min T.C. pairs of \( S \) and \( \mathcal{S} \) are identical, say \( \beta_j, \beta_h \) with \( j < h \). Let \( S_1 \) and \( \mathcal{S}_1 \) be the H.T. sequences obtained by one H.T.-step from \( S \) and \( \mathcal{S} \) respectively.

**Case 1:** \( h < i \), then \( \beta_j \) and \( \beta_h \) both lie in A. Then \( S_1 = A' \ w \ B \ C \) and \( \mathcal{S}_1 = A' \ B' \ w \ C \) with \( A' = \beta_{i-1} \beta_{j-1} \cdots \beta_{h-1} \beta_j \beta_h \beta_{i+1} \cdots \beta_{i-1} \). \( S_1 \) satisfies 1), 2) and 3) and hence \( S_1 \) and \( \mathcal{S}_1 \) are H.T.-equivalent. So \( S \) and \( \mathcal{S} \) are H.T.-equivalent. Contradiction.

**Case 2:** \( i+1 \leq j < h \leq i+k \), then \( \beta_j \) and \( \beta_h \) both lie in B. Then \( S_1 = A \ w \ B' \ C \) and \( \mathcal{S}_1 = A \ B' \ w \ C \) with \( B' = \beta_{i+1} \cdots \beta_{j-1} \beta_{j+1} \cdots \beta_{h-1} \beta_j \beta_h \beta_{i+1} \cdots \beta_{i+k} \). Let \( u = \beta_j + \beta_h \). If \( u < w \) then \( S_1 \) satisfies 1), 2) and 3) and hence \( S_1 \) and \( \mathcal{S}_1 \) are equivalent. If \( u \geq w \) then \( u > w \) by assumption 3). Hence \( S_1 = A \ w \ B \ u \ B_2 \ C \) satisfies 1), 2) and 3) with \( u \) instead of \( w \) and \( B_2 \) instead of \( B \) [it satisfies 3] because of Lemma 2]. Hence \( S_1 \) is H.T.-equivalent to \( A \ w \ B_1 \ B_2 \ u \ C \) and this sequence is equivalent to \( A \ B_1 \ B_2 \ w \ u \ C \). The same argument shows
that \( \tilde{S} \) is H.T.-equivalent to \( A B_1 B_2 \ldots w u C \).

The other cases (\( \beta_j \) in A, \( \beta_h \) in B or \( \beta_j \) in A, \( \beta_h \) in C or ...) are treated analogously.

The equivalence of algorithms H.T. and M.T. is a direct consequence of Lemma 3. Let \( \alpha_1, \alpha_2, \ldots, \alpha_p \) be an initial sequence consisting of squares and let \( S_{H.T.}^{(m)} \) and \( S_{M.T.}^{(m)} \) be the sequences obtained from it by \( m \) H.T.-steps or M.T.-steps respectively.

For \( m = 0 \) \( S_{H.T.}^{(0)} = S_{M.T.}^{(0)} \). Suppose now that \( S_{H.T.}^{(m)} \) and \( S_{M.T.}^{(m)} \) are H.T.-equivalent. We want to show that \( S_{H.T.}^{(m+1)} \) and \( S_{M.T.}^{(m+1)} \) are H.T.-equivalent. Let \( S \) be the sequence obtained by one H.T.-step from \( S_{M.T.}^{(m)} \), then \( S = A \wedge B \wedge C \) with \( w \) being the node formed by that H.T.-step and \( v < w \) for all \( v \) in \( B \) and \( v > w \) for the first node of \( C \). \( S \) satisfies 1), 2) and 3) of Lemma 3 (Lemma 2 implies 3)) and \( \tilde{S} = A B \wedge w C = S_{M.T.}^{(m+1)} \). Hence \( S \) and \( S_{M.T.}^{(m+1)} \) are H.T.-equivalent. Furthermore \( S \) and \( S_{H.T.}^{(m+1)} \) are H.T.-equivalent by definition of H.T.-equivalence. This shows that \( S_{H.T.}^{(m+1)} \) and \( S_{M.T.}^{(m+1)} \) are H.T.-equivalent and proves:

**Theorem:** Algorithms H.T., M.T. and C.W. perform exactly the same sequence of combinations.

In particular, the Hu/Tucker- and Carstens/Wachs-algorithms manipulate their data structure in an isomorphic way. As an immediate consequence we obtain a new correctness proof for the Hu/Tucker-algorithm.
Bibliography


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Figure 1: Algorithm G.W. on sequence 5, 3, 3, 2, 2, 2, 3, 4

Figure 2: The untaugled version of the trees in Figure 1, 3 and 4.
Figure 3: Algorithm H.T. on sequence $5, 3, 1, 2, 6, 8, 4$.

Figure 4: Algorithm H.T. on sequence $5, 3, 2, 2, 21, 2, 13, 8$.