Formalisation of Ground Resolution and CDCL in Isabelle/HOL

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Contents

theory *Prop-Abstract-Transformation*
imports *Entailment-Definition.Prop-Logic Weidenbach-Book-Base.Wellfounded-More*

begin

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

0.1 Rewrite Systems and Properties

0.1.1 Lifting of Rewrite Rules

We can lift a rewrite relation r over a full1 formula: the relation *r* works on terms, while *propo-rew-step* works on formulas.

inductive *propo-rew-step* :: $('v\ propo \Rightarrow 'v\ propo \Rightarrow bool) \Rightarrow 'v\ propo \Rightarrow 'v\ propo \Rightarrow bool$
  for $r :: 'v\ propo \Rightarrow 'v\ propo \Rightarrow bool$ **where**

global-rel: $r\ \varphi\ \psi \Longrightarrow$ *propo-rew-step* $r\ \varphi\ \psi$ |
propo-rew-one-step-lift: *propo-rew-step* $r\ \varphi\ \varphi' \Longrightarrow$ *wf-conn* $c\ (\psi s\ @\ \varphi\ \#\ \psi s')$
$\Longrightarrow$ *propo-rew-step* r (*conn* $c\ (\psi s\ @\ \varphi\ \#\ \psi s')$) (*conn* $c\ (\psi s\ @\ \varphi'\#\ \psi s')$)

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ', then there are two subformulas ψ in φ and ψ' in φ', ψ' is the result of the rewriting of r on ψ.

This lemma is only a health condition:

lemma *propo-rew-step-subformula-imp*:
shows *propo-rew-step* $r\ \varphi\ \varphi' \Longrightarrow \exists\ \psi\ \psi'.\ \psi \preceq \varphi \land \psi' \preceq \varphi' \land r\ \psi\ \psi'$
⟨*proof*⟩

The converse is moreover true: if there is a ψ and ψ', then every formula φ containing ψ, can be rewritten into a formula φ', such that it contains φ'.

lemma *propo-rew-step-subformula-rec*:
fixes $\psi\ \psi'\ \varphi$ :: *'v propo*
shows $\psi \preceq \varphi \Longrightarrow r\ \psi\ \psi' \Longrightarrow (\exists\varphi'.\ \psi' \preceq \varphi' \land$ *propo-rew-step* $r\ \varphi\ \varphi')$
⟨*proof*⟩

lemma *propo-rew-step-subformula*:
$(\exists\psi\ \psi'.\ \psi \preceq \varphi \land r\ \psi\ \psi') \longleftrightarrow (\exists\varphi'.$ *propo-rew-step* $r\ \varphi\ \varphi')$
⟨*proof*⟩

lemma *consistency-decompose-into-list*:
assumes *wf*: *wf-conn c l* **and** *wf'*: *wf-conn c l'*
and *same*: $\forall n.\ A \models l\ !\ n \longleftrightarrow (A \models l'\ !\ n)$
shows $A \models$ *conn c l* $\longleftrightarrow A \models$ *conn c l'*
⟨*proof*⟩

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step* $r\ \varphi\ \varphi'$ means that we rewrite ψ inside φ (ie at a path p) into ψ'.

lemma *propo-rew-step-rewrite*:
fixes $\varphi\ \varphi'$ :: *'v propo* **and** r :: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool*
assumes *propo-rew-step* $r\ \varphi\ \varphi'$
shows $\exists\psi\ \psi'\ p.\ r\ \psi\ \psi' \land$ *path-to* $p\ \varphi\ \psi \land$ *replace-at* $p\ \varphi\ \psi' = \varphi'$
⟨*proof*⟩

0.1.2 Consistency Preservation

We define *preserve-models*: it means that a relation preserves consistency.

definition *preserve-models* **where**
preserve-models $r \longleftrightarrow (\forall\varphi\ \psi.\ r\ \varphi\ \psi \longrightarrow (\forall A.\ A \models \varphi \longleftrightarrow A \models \psi))$

lemma *propo-rew-step-preservers-val-explicit*:
propo-rew-step $r\ \varphi\ \psi \Longrightarrow$ *preserve-models* $r \Longrightarrow$ *propo-rew-step* $r\ \varphi\ \psi \Longrightarrow (\forall A.\ A \models\varphi \longleftrightarrow A\models\psi)$
⟨*proof*⟩

lemma *propo-rew-step-preservers-val'*:
assumes *preserve-models r*
shows *preserve-models* (*propo-rew-step r*)
⟨*proof*⟩

lemma *preserve-models-OO*[*intro*]:
preserve-models f $\Longrightarrow$ *preserve-models g* $\Longrightarrow$ *preserve-models* (*f OO g*)
⟨*proof*⟩

lemma *star-consistency-preservation-explicit*:
assumes (*propo-rew-step r*)^** φ ψ **and** *preserve-models r*
shows $\forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
⟨*proof*⟩

lemma *star-consistency-preservation*:
preserve-models r $\Longrightarrow$ *preserve-models* (*propo-rew-step r*)^**
⟨*proof*⟩

0.1.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

lemma *full-ropo-rew-step-preservers-val*[*simp*]:
preserve-models r $\Longrightarrow$ *preserve-models* (*full* (*propo-rew-step r*))
⟨*proof*⟩

lemma *full-propo-rew-step-subformula*:
full (*propo-rew-step r*) φ' φ $\Longrightarrow \neg(\exists\ \psi\ \psi'.\ \psi \preceq \varphi \land r\ \psi\ \psi')$
⟨*proof*⟩

0.2 Transformation testing

0.2.1 Definition and first Properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: (*'a propo* $\Rightarrow$ *bool*) $\Rightarrow$ *'a propo* $\Rightarrow$ *bool* **where**
all-subformula-st test-symb $\varphi \equiv \forall\psi.\ \psi \preceq \varphi \longrightarrow$ *test-symb* ψ

lemma *test-symb-imp-all-subformula-st*[*simp*]:
test-symb FT $\Longrightarrow$ *all-subformula-st test-symb FT*
test-symb FF $\Longrightarrow$ *all-subformula-st test-symb FF*
test-symb (*FVar x*) $\Longrightarrow$ *all-subformula-st test-symb* (*FVar x*)
⟨*proof*⟩

lemma *all-subformula-st-test-symb-true-phi*:
all-subformula-st test-symb φ $\Longrightarrow$ *test-symb* φ
⟨*proof*⟩

lemma *all-subformula-st-decomp-imp*:

wf-conn c l $\Longrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall\varphi\in$ *set l. all-subformula-st test-symb* φ))
$\Longrightarrow$ *all-subformula-st test-symb* (*conn c l*)
⟨*proof*⟩

To ease the finding of proofs, we give some explicit theorem about the decomposition.

lemma *all-subformula-st-decomp-rec*:
all-subformula-st test-symb (*conn c l*) $\Longrightarrow$ *wf-conn c l*
$\Longrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall\varphi\in$ *set l. all-subformula-st test-symb* φ))
⟨*proof*⟩

lemma *all-subformula-st-decomp*:
fixes *c* :: *'v connective* **and** *l* :: *'v propo list*
assumes *wf-conn c l*
shows *all-subformula-st test-symb* (*conn c l*)
$\longleftrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall\varphi\in$ *set l. all-subformula-st test-symb* φ))
⟨*proof*⟩

lemma *helper-fact*: $c \in$ *binary-connectives* $\longleftrightarrow$ (c = *COr* $\vee$ c = *CAnd* $\vee$ c = *CEq* $\vee$ c = *CImp*)
⟨*proof*⟩
lemma *all-subformula-st-decomp-explicit*[*simp*]:
fixes φ ψ :: *'v propo*
shows *all-subformula-st test-symb* (*FAnd* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FAnd* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FOr* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FOr* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FNot* φ)
$\longleftrightarrow$ (*test-symb* (*FNot* φ) $\wedge$ *all-subformula-st test-symb* φ)
and *all-subformula-st test-symb* (*FEq* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FEq* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FImp* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FImp* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
⟨*proof*⟩

As *all-subformula-st* tests recursively, the function is true on every subformula.

lemma *subformula-all-subformula-st*:
$\psi \preceq \varphi \Longrightarrow$ *all-subformula-st test-symb* $\varphi \Longrightarrow$ *all-subformula-st test-symb* ψ
⟨*proof*⟩

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation r: if we assume that if every time *test-symb* is true, then a r can be applied, finally as long as $\neg$ *all-subformula-st test-symb* φ, then something can be rewritten in φ.

lemma *no-test-symb-step-exists*:
fixes *r*:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** x :: *'v*
and φ :: *'v propo*
assumes
test-symb-false-nullary: $\forall x.$ *test-symb FF* $\wedge$ *test-symb FT* $\wedge$ *test-symb* (*FVar x*) **and**
$\forall\varphi'. \varphi' \preceq \varphi \longrightarrow (\neg$*test-symb* $\varphi') \longrightarrow (\exists\ \psi.\ r\ \varphi'\ \psi)$ **and**
$\neg$ *all-subformula-st test-symb* φ
shows $\exists\psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi'$
⟨*proof*⟩

0.2.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the same property, with changes in the assumptions.

The assumption $\forall\,\varphi'\,\psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow \textit{all-subformula-st test-symb}\ \varphi' \longrightarrow \textit{all-subformula-st test-symb}\ \psi$ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r: we have to add the assumption that rewriting inside does not mess up the term: $\forall\, c\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow \textit{propo-rew-step}\ r\ \varphi\ \varphi' \longrightarrow \textit{wf-conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow \textit{test-symb}\ \varphi' \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$

Invariant while lifting of the Rewriting Relation

The condition $\varphi \preceq \Phi$ (that will by used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ, we do not have to care about equivalence symbols in the two previous assumptions.

lemma *propo-rew-step-inv-stay'*:
  fixes r:: $'v\ \textit{propo} \Rightarrow\ 'v\ \textit{propo} \Rightarrow \textit{bool}$ **and** *test-symb*:: $'v\ \textit{propo} \Rightarrow \textit{bool}$ **and** x :: $'v$
  and $\varphi\ \psi\ \Phi$:: $'v\ \textit{propo}$
  assumes H: $\forall\,\varphi'\,\psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow \textit{all-subformula-st test-symb}\ \varphi'$
    $\longrightarrow \textit{all-subformula-st test-symb}\ \psi$
  and H': $\forall\,(c\text{:: }'v\ \textit{connective})\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow \textit{propo-rew-step}\ r\ \varphi\ \varphi'$
    $\longrightarrow \textit{wf-conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow \textit{test-symb}\ \varphi'$
    $\longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$ **and**
    propo-rew-step $r\ \varphi\ \psi$ **and**
    $\varphi \preceq \Phi$ **and**
    all-subformula-st test-symb φ
  shows *all-subformula-st test-symb* ψ
  ⟨*proof*⟩

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

lemma *propo-rew-step-inv-stay*:
  fixes r:: $'v\ \textit{propo} \Rightarrow\ 'v\ \textit{propo} \Rightarrow \textit{bool}$ **and** *test-symb*:: $'v\ \textit{propo} \Rightarrow \textit{bool}$ **and** x :: $'v$
  and $\varphi\ \psi$ :: $'v\ \textit{propo}$
  assumes
    H: $\forall\,\varphi'\ \psi.\ r\ \varphi'\ \psi \longrightarrow \textit{all-subformula-st test-symb}\ \varphi' \longrightarrow \textit{all-subformula-st test-symb}\ \psi$ **and**
    H': $\forall\,(c\text{:: }'v\ \textit{connective})\ \xi\ \varphi\ \xi'\ \varphi'.\ \textit{wf-conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi'))$
      $\longrightarrow \textit{test-symb}\ \varphi' \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$ **and**
    propo-rew-step $r\ \varphi\ \psi$ **and**
    all-subformula-st test-symb φ
  shows *all-subformula-st test-symb* ψ
  ⟨*proof*⟩

The lemmas can be lifted to *propo-rew-step* $r^{\downarrow}$ instead of *propo-rew-step*

Invariant after all Rewriting

lemma *full-propo-rew-step-inv-stay-with-inc*:
  fixes r:: $'v\ \textit{propo} \Rightarrow\ 'v\ \textit{propo} \Rightarrow \textit{bool}$ **and** *test-symb*:: $'v\ \textit{propo} \Rightarrow \textit{bool}$ **and** x :: $'v$

and $\varphi\ \psi$ :: $'v$ *propo*
assumes
H: $\forall\ \varphi\ \psi.$ *propo-rew-step r* $\varphi\ \psi \longrightarrow$ *all-subformula-st test-symb* φ
$\longrightarrow$ *all-subformula-st test-symb* ψ **and**
H′: $\forall\,(c$:: $'v$ *connective*$)\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow$ *propo-rew-step r* $\varphi\ \varphi'$
$\longrightarrow$ *wf-conn c* $(\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow$ *test-symb* φ'
$\longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$) **and**
$\varphi \preceq \Phi$ **and**
full: *full* (*propo-rew-step r*) $\varphi\ \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
⟨*proof*⟩

lemma *full-propo-rew-step-inv-stay′*:
fixes *r*:: $'v$ *propo* ⇒ $'v$ *propo* ⇒ *bool* **and** *test-symb*:: $'v$ *propo* ⇒ *bool* **and** *x* :: $'v$
and $\varphi\ \psi$ :: $'v$ *propo*
assumes
H: $\forall\ \varphi\ \psi.$ *propo-rew-step r* $\varphi\ \psi \longrightarrow$ *all-subformula-st test-symb* φ
$\longrightarrow$ *all-subformula-st test-symb* ψ **and**
H′: $\forall\,(c$:: $'v$ *connective*$)\ \xi\ \varphi\ \xi'\ \varphi'.$ *propo-rew-step r* $\varphi\ \varphi' \longrightarrow$ *wf-conn c* $(\xi\ @\ \varphi\ \#\ \xi')$
$\longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow$ *test-symb* $\varphi' \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$) **and**
full: *full* (*propo-rew-step r*) $\varphi\ \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
⟨*proof*⟩

lemma *full-propo-rew-step-inv-stay*:
fixes *r*:: $'v$ *propo* ⇒ $'v$ *propo* ⇒ *bool* **and** *test-symb*:: $'v$ *propo* ⇒ *bool* **and** *x* :: $'v$
and $\varphi\ \psi$ :: $'v$ *propo*
assumes
H: $\forall\varphi\ \psi.\ r\ \varphi\ \psi \longrightarrow$ *all-subformula-st test-symb* $\varphi \longrightarrow$ *all-subformula-st test-symb* ψ **and**
H′: $\forall\,(c$:: $'v$ *connective*$)\ \xi\ \varphi\ \xi'\ \varphi'.$ *wf-conn c* $(\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi\ \#\ \xi')$)
$\longrightarrow$ *test-symb* $\varphi' \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$) **and**
full: *full* (*propo-rew-step r*) $\varphi\ \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
⟨*proof*⟩

lemma *full-propo-rew-step-inv-stay-conn*:
fixes *r*:: $'v$ *propo* ⇒ $'v$ *propo* ⇒ *bool* **and** *test-symb*:: $'v$ *propo* ⇒ *bool* **and** *x* :: $'v$
and $\varphi\ \psi$ :: $'v$ *propo*
assumes
H: $\forall\varphi\ \psi.\ r\ \varphi\ \psi \longrightarrow$ *all-subformula-st test-symb* $\varphi \longrightarrow$ *all-subformula-st test-symb* ψ **and**
H′: $\forall\,(c$:: $'v$ *connective*$)\ l\ l'.$ *wf-conn c l* $\longrightarrow$ *wf-conn c l′*
$\longrightarrow$ (*test-symb* (*conn c l*) $\longleftrightarrow$ *test-symb* (*conn c l′*)) **and**
full: *full* (*propo-rew-step r*) $\varphi\ \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
⟨*proof*⟩

end
theory *Prop-Normalisation*
imports *Entailment-Definition.Prop-Logic Prop-Abstract-Transformation Nested-Multisets-Ordinals.Multiset-More*
begin

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

0.3 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation seperately.

0.3.1 Elimination of the Equivalences

The first transformation consists in removing every equivalence symbol.

inductive *elim-equiv* :: *'v propo* ⇒ *'v propo* ⇒ *bool* **where**
elim-equiv[*simp*]: *elim-equiv* (*FEq* φ ψ) (*FAnd* (*FImp* φ ψ) (*FImp* ψ φ))

lemma *elim-equiv-transformation-consistent*:
A ⊨ *FEq* φ ψ ⟷ *A* ⊨ *FAnd* (*FImp* φ ψ) (*FImp* ψ φ)
⟨*proof*⟩

lemma *elim-equiv-explicit*: *elim-equiv* φ ψ ⟹ ∀ *A*. *A* ⊨ φ ⟷ *A* ⊨ ψ
⟨*proof*⟩

lemma *elim-equiv-consistent*: *preserve-models elim-equiv*
⟨*proof*⟩

lemma *elimEquv-lifted-consistant*:
preserve-models (*full* (*propo-rew-step elim-equiv*))
⟨*proof*⟩

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

fun *no-equiv-symb* :: *'v propo* ⇒ *bool* **where**
no-equiv-symb (*FEq* - -) = *False* |
no-equiv-symb - = *True*

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

lemma *no-equiv-symb-conn-characterization*[*simp*]:
fixes *c* :: *'v connective* **and** *l* :: *'v propo list*
assumes *wf*: *wf-conn c l*
shows *no-equiv-symb* (*conn c l*) ⟷ *c* ≠ *CEq*
⟨*proof*⟩

definition *no-equiv* **where** *no-equiv* = *all-subformula-st no-equiv-symb*

lemma *no-equiv-eq*[*simp*]:
fixes φ ψ :: *'v propo*
shows
¬*no-equiv* (*FEq* φ ψ)
no-equiv FT
no-equiv FF
⟨*proof*⟩

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

lemma *all-subformula-st-decomp-explicit-no-equiv*[*iff*]:
fixes φ ψ :: *'v propo*
shows
  no-equiv (*FNot* φ) $\longleftrightarrow$ *no-equiv* φ
  no-equiv (*FAnd* φ ψ) $\longleftrightarrow$ (*no-equiv* φ $\wedge$ *no-equiv* ψ)
  no-equiv (*FOr* φ ψ) $\longleftrightarrow$ (*no-equiv* φ $\wedge$ *no-equiv* ψ)
  no-equiv (*FImp* φ ψ) $\longleftrightarrow$ (*no-equiv* φ $\wedge$ *no-equiv* ψ)
  ⟨*proof*⟩

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

lemma *no-equiv-elim-equiv-step*:
  fixes φ :: *'v propo*
  assumes *no-equiv*: $\neg$ *no-equiv* φ
  shows $\exists\,\psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elim-equiv* ψ ψ'
⟨*proof*⟩

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:
  full (*propo-rew-step elim-equiv*) φ ψ $\Longrightarrow$ *no-equiv* ψ
  ⟨*proof*⟩

0.3.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* :: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **where**
[*simp*]: *elim-imp* (*FImp* φ ψ) (*FOr* (*FNot* φ) ψ)

lemma *elim-imp-transformation-consistent*:
  $A \models$ *FImp* φ ψ $\longleftrightarrow$ $A \models$ *FOr* (*FNot* φ) ψ
  ⟨*proof*⟩

lemma *elim-imp-explicit*: *elim-imp* φ ψ $\Longrightarrow$ $\forall\,A.\ A \models \varphi \longleftrightarrow A \models \psi$
  ⟨*proof*⟩

lemma *elim-imp-consistent*: *preserve-models elim-imp*
  ⟨*proof*⟩

lemma *elim-imp-lifted-consistant*:
  preserve-models (*full* (*propo-rew-step elim-imp*))
  ⟨*proof*⟩

fun *no-imp-symb* **where**
no-imp-symb (*FImp* - -) = *False* |
no-imp-symb - = *True*

lemma *no-imp-symb-conn-characterization*:
  wf-conn c l $\Longrightarrow$ *no-imp-symb* (*conn c l*) $\longleftrightarrow$ $c \neq$ *CImp*
  ⟨*proof*⟩

definition *no-imp* **where** *no-imp* $\equiv$ *all-subformula-st no-imp-symb*

declare *no-imp-def*[*simp*]

lemma *no-imp-Imp*[*simp*]:
$\neg$*no-imp* (*FImp* φ ψ)
no-imp FT
no-imp FF
$\langle$*proof*$\rangle$

lemma *all-subformula-st-decomp-explicit-imp*[*simp*]:
fixes φ ψ :: *'v propo*
shows
no-imp (*FNot* φ) $\longleftrightarrow$ *no-imp* φ
no-imp (*FAnd* φ ψ) $\longleftrightarrow$ (*no-imp* φ $\wedge$ *no-imp* ψ)
no-imp (*FOr* φ ψ) $\longleftrightarrow$ (*no-imp* φ $\wedge$ *no-imp* ψ)
$\langle$*proof*$\rangle$

Invariant of the *elim-imp* transformation

lemma *elim-imp-no-equiv*:
elim-imp φ ψ $\Longrightarrow$ *no-equiv* φ $\Longrightarrow$ *no-equiv* ψ
$\langle$*proof*$\rangle$

lemma *elim-imp-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elim-imp*) φ ψ **and** *no-equiv* φ
shows *no-equiv* ψ
$\langle$*proof*$\rangle$

lemma *no-no-imp-elim-imp-step-exists*:
fixes φ :: *'v propo*
assumes *no-equiv*: $\neg$ *no-imp* φ
shows $\exists\psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elim-imp* ψ ψ'
$\langle$*proof*$\rangle$

lemma *no-imp-full-propo-rew-step-elim-imp*: *full* (*propo-rew-step elim-imp*) φ ψ $\Longrightarrow$ *no-imp* ψ
$\langle$*proof*$\rangle$

0.3.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the "commutative" transformation. The latter is implicit in the book.

inductive *elimTB* **where**
ElimTB1: *elimTB* (*FAnd* φ *FT*) φ |
ElimTB1': *elimTB* (*FAnd FT* φ) φ |

ElimTB2: *elimTB* (*FAnd* φ *FF*) *FF* |
ElimTB2': *elimTB* (*FAnd FF* φ) *FF* |

ElimTB3: *elimTB* (*FOr* φ *FT*) *FT* |
ElimTB3': *elimTB* (*FOr FT* φ) *FT* |

ElimTB4: *elimTB* (*FOr* φ *FF*) φ |
ElimTB4': *elimTB* (*FOr FF* φ) φ |

ElimTB5: *elimTB* (*FNot FT*) *FF* |
ElimTB6: *elimTB* (*FNot FF*) *FT*

lemma *elimTB-consistent*: *preserve-models elimTB*
⟨*proof*⟩

inductive *no-T-F-symb* :: *'v propo* ⇒ *bool* **where**
no-T-F-symb-comp: *c* ≠ *CF* ⟹ *c* ≠ *CT* ⟹ *wf-conn c l* ⟹ (∀ *φ* ∈ *set l*. *φ* ≠ *FT* ∧ *φ* ≠ *FF*)
⟹ *no-T-F-symb* (*conn c l*)

lemma *wf-conn-no-T-F-symb-iff*[*simp*]:
wf-conn c ψs ⟹
no-T-F-symb (*conn c ψs*) ⟷ (*c* ≠ *CF* ∧ *c* ≠ *CT* ∧ (∀ *ψ*∈*set ψs*. *ψ* ≠ *FF* ∧ *ψ* ≠ *FT*))
⟨*proof*⟩

lemma *wf-conn-no-T-F-symb-iff-explicit*[*simp*]:
no-T-F-symb (*FAnd φ ψ*) ⟷ (∀ *χ* ∈ *set* [*φ*, *ψ*]. *χ* ≠ *FF* ∧ *χ* ≠ *FT*)
no-T-F-symb (*FOr φ ψ*) ⟷ (∀ *χ* ∈ *set* [*φ*, *ψ*]. *χ* ≠ *FF* ∧ *χ* ≠ *FT*)
no-T-F-symb (*FEq φ ψ*) ⟷ (∀ *χ* ∈ *set* [*φ*, *ψ*]. *χ* ≠ *FF* ∧ *χ* ≠ *FT*)
no-T-F-symb (*FImp φ ψ*) ⟷ (∀ *χ* ∈ *set* [*φ*, *ψ*]. *χ* ≠ *FF* ∧ *χ* ≠ *FT*)
⟨*proof*⟩

lemma *no-T-F-symb-false*[*simp*]:
fixes *c* :: *'v connective*
shows
¬*no-T-F-symb* (*FT* :: *'v propo*)
¬*no-T-F-symb* (*FF* :: *'v propo*)
⟨*proof*⟩

lemma *no-T-F-symb-bool*[*simp*]:
fixes *x* :: *'v*
shows *no-T-F-symb* (*FVar x*)
⟨*proof*⟩

lemma *no-T-F-symb-fnot-imp*:
¬*no-T-F-symb* (*FNot φ*) ⟹ *φ* = *FT* ∨ *φ* = *FF*
⟨*proof*⟩

lemma *no-T-F-symb-fnot*[*simp*]:
no-T-F-symb (*FNot φ*) ⟷ ¬(*φ* = *FT* ∨ *φ* = *FF*)
⟨*proof*⟩

Actually it is not possible to remover every *FT* and *FF*: if the formula is equal to true or false, we can not remove it.

inductive *no-T-F-symb-except-toplevel* **where**
no-T-F-symb-except-toplevel-true[*simp*]: *no-T-F-symb-except-toplevel FT* |
no-T-F-symb-except-toplevel-false[*simp*]: *no-T-F-symb-except-toplevel FF* |
noTrue-no-T-F-symb-except-toplevel[*simp*]: *no-T-F-symb φ* ⟹ *no-T-F-symb-except-toplevel φ*

lemma *no-T-F-symb-except-toplevel-bool*:
fixes *x* :: *'v*
shows *no-T-F-symb-except-toplevel* (*FVar x*)
⟨*proof*⟩

lemma *no-T-F-symb-except-toplevel-not-decom*:
$\varphi \neq FT \Longrightarrow \varphi \neq FF \Longrightarrow$ *no-T-F-symb-except-toplevel* (*FNot* φ)
⟨*proof*⟩

lemma *no-T-F-symb-except-toplevel-bin-decom*:
fixes φ ψ :: *'v propo*
assumes $\varphi \neq FT$ **and** $\varphi \neq FF$ **and** $\psi \neq FT$ **and** $\psi \neq FF$
and *c*: *c*$\in$ *binary-connectives*
shows *no-T-F-symb-except-toplevel* (*conn c* [φ, ψ])
⟨*proof*⟩

lemma *no-T-F-symb-except-toplevel-if-is-a-true-false*:
fixes *l* :: *'v propo list* **and** *c* :: *'v connective*
assumes *corr*: *wf-conn c l*
and $FT \in$ *set l* $\vee$ $FF \in$ *set l*
shows $\neg$*no-T-F-symb-except-toplevel* (*conn c l*)
⟨*proof*⟩

lemma *no-T-F-symb-except-top-level-false-example*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$
shows
$\neg$ *no-T-F-symb-except-toplevel* (*FAnd* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FOr* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FImp* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FEq* φ ψ)
⟨*proof*⟩

lemma *no-T-F-symb-except-top-level-false-not*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \vee \varphi = FF$
shows
$\neg$ *no-T-F-symb-except-toplevel* (*FNot* φ)
⟨*proof*⟩

This is the local extension of *no-T-F-symb-except-toplevel*.

definition *no-T-F-except-top-level* **where**
no-T-F-except-top-level $\equiv$ *all-subformula-st no-T-F-symb-except-toplevel*

This is another property we will use. While this version might seem to be the one we want to prove, it is not since *FT* can not be reduced.

definition *no-T-F* **where**
no-T-F $\equiv$ *all-subformula-st no-T-F-symb*

lemma *no-T-F-except-top-level-false*:
fixes *l* :: *'v propo list* **and** *c* :: *'v connective*
assumes *wf-conn c l*
and $FT \in$ *set l* $\vee$ $FF \in$ *set l*
shows $\neg$*no-T-F-except-top-level* (*conn c l*)
⟨*proof*⟩

lemma *no-T-F-except-top-level-false-example*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$

shows
$\neg$*no-T-F-except-top-level* (*FAnd* φ ψ)
$\neg$*no-T-F-except-top-level* (*FOr* φ ψ)
$\neg$*no-T-F-except-top-level* (*FEq* φ ψ)
$\neg$*no-T-F-except-top-level* (*FImp* φ ψ)
$\langle$*proof*$\rangle$

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:
no-T-F-symb-except-toplevel $\varphi \Longrightarrow \varphi \neq$ *FF* $\Longrightarrow \varphi \neq$ *FT* $\Longrightarrow$ *no-T-F-symb* φ
$\langle$*proof*$\rangle$

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:
no-T-F-except-top-level $\varphi \Longrightarrow \varphi \neq$ *FF* $\Longrightarrow \varphi \neq$ *FT* $\Longrightarrow$ *no-T-F* φ
$\langle$*proof*$\rangle$

lemma *no-T-F-no-T-F-except-top-level*:
no-T-F $\varphi \Longrightarrow$ *no-T-F-except-top-level* φ
$\langle$*proof*$\rangle$

lemma *no-T-F-except-top-level-simp*[*simp*]: *no-T-F-except-top-level FF no-T-F-except-top-level FT*
$\langle$*proof*$\rangle$

lemma *no-T-F-no-T-F-except-top-level'*[*simp*]:
no-T-F-except-top-level $\varphi \longleftrightarrow (\varphi =$ *FF* $\lor\ \varphi =$ *FT* $\lor$ *no-T-F* $\varphi)$
$\langle$*proof*$\rangle$

lemma *no-T-F-bin-decomp*[*simp*]:
assumes *c*: $c \in$ *binary-connectives*
shows *no-T-F* (*conn c* [φ, ψ]) $\longleftrightarrow$ (*no-T-F* φ $\land$ *no-T-F* ψ)
$\langle$*proof*$\rangle$

lemma *no-T-F-bin-decomp-expanded*[*simp*]:
assumes *c*: $c =$ *CAnd* $\lor\ c =$ *COr* $\lor\ c =$ *CEq* $\lor\ c =$ *CImp*
shows *no-T-F* (*conn c* [φ, ψ]) $\longleftrightarrow$ (*no-T-F* φ $\land$ *no-T-F* ψ)
$\langle$*proof*$\rangle$

lemma *no-T-F-comp-expanded-explicit*[*simp*]:
fixes φ ψ :: *'v propo*
shows
no-T-F (*FAnd* φ ψ) $\longleftrightarrow$ (*no-T-F* φ $\land$ *no-T-F* ψ)
no-T-F (*FOr* φ ψ) $\longleftrightarrow$ (*no-T-F* φ $\land$ *no-T-F* ψ)
no-T-F (*FEq* φ ψ) $\longleftrightarrow$ (*no-T-F* φ $\land$ *no-T-F* ψ)
no-T-F (*FImp* φ ψ) $\longleftrightarrow$ (*no-T-F* φ $\land$ *no-T-F* ψ)
$\langle$*proof*$\rangle$

lemma *no-T-F-comp-not*[*simp*]:
fixes φ ψ :: *'v propo*
shows *no-T-F* (*FNot* φ) $\longleftrightarrow$ *no-T-F* φ
$\langle$*proof*$\rangle$

lemma *no-T-F-decomp*:
fixes φ ψ :: *'v propo*
assumes φ: *no-T-F* (*FAnd* φ ψ) $\lor$ *no-T-F* (*FOr* φ ψ) $\lor$ *no-T-F* (*FEq* φ ψ) $\lor$ *no-T-F* (*FImp* φ ψ)
shows *no-T-F* ψ **and** *no-T-F* φ

⟨*proof*⟩

lemma *no-T-F-decomp-not*:
fixes φ :: *'v propo*
assumes φ: *no-T-F* (*FNot* φ)
shows *no-T-F* φ
⟨*proof*⟩

lemma *no-T-F-symb-except-toplevel-step-exists*:
fixes φ ψ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ
shows $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists\, \psi'$. *elimTB* ψ ψ'
⟨*proof*⟩

lemma *no-T-F-except-top-level-rew*:
fixes φ :: *'v propo*
assumes *noTB*: $\neg$ *no-T-F-except-top-level* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists\, \psi\ \psi'.\ \psi \preceq \varphi \land$ *elimTB* ψ ψ'
⟨*proof*⟩

lemma *elimTB-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elimTB*) φ ψ
and *no-equiv* φ **and** *no-imp* φ
shows *no-equiv* ψ **and** *no-imp* ψ
⟨*proof*⟩

lemma *elimTB-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ **and** *full* (*propo-rew-step elimTB*) φ ψ
shows *no-T-F-except-top-level* ψ
⟨*proof*⟩

0.3.4 PushNeg

Push the negation inside the formula, until the litteral.

inductive *pushNeg* **where**
PushNeg1[*simp*]: *pushNeg* (*FNot* (*FAnd* φ ψ)) (*FOr* (*FNot* φ) (*FNot* ψ)) |
PushNeg2[*simp*]: *pushNeg* (*FNot* (*FOr* φ ψ)) (*FAnd* (*FNot* φ) (*FNot* ψ)) |
PushNeg3[*simp*]: *pushNeg* (*FNot* (*FNot* φ)) φ

lemma *pushNeg-transformation-consistent*:
$A \models$ *FNot* (*FAnd* φ ψ) $\longleftrightarrow A \models$ (*FOr* (*FNot* φ) (*FNot* ψ))
$A \models$ *FNot* (*FOr* φ ψ) $\longleftrightarrow A \models$ (*FAnd* (*FNot* φ) (*FNot* ψ))
$A \models$ *FNot* (*FNot* φ) $\longleftrightarrow A \models \varphi$
⟨*proof*⟩

lemma *pushNeg-explicit*: *pushNeg* φ $\psi \Longrightarrow \forall\, A.\ A \models \varphi \longleftrightarrow A \models \psi$
⟨*proof*⟩

lemma *pushNeg-consistent*: *preserve-models pushNeg*
⟨*proof*⟩

lemma *pushNeg-lifted-consistant*:
preserve-models (*full* (*propo-rew-step pushNeg*))
⟨*proof*⟩

fun *simple* **where**
simple FT = *True* |
simple FF = *True* |
simple (*FVar* -) = *True* |
simple - = *False*

lemma *simple-decomp*:
simple $\varphi \longleftrightarrow (\varphi = \mathit{FT} \vee \varphi = \mathit{FF} \vee (\exists x.\ \varphi = \mathit{FVar}\ x))$
⟨*proof*⟩

lemma *subformula-conn-decomp-simple*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *s*: *simple* ψ
shows $\varphi \preceq \mathit{FNot}\ \psi \longleftrightarrow (\varphi = \mathit{FNot}\ \psi \vee \varphi = \psi)$
⟨*proof*⟩

lemma *subformula-conn-decomp-explicit*[*simp*]:
fixes φ :: *'v propo* **and** *x* :: *'v*
shows
$\varphi \preceq \mathit{FNot}\ \mathit{FT} \longleftrightarrow (\varphi = \mathit{FNot}\ \mathit{FT} \vee \varphi = \mathit{FT})$
$\varphi \preceq \mathit{FNot}\ \mathit{FF} \longleftrightarrow (\varphi = \mathit{FNot}\ \mathit{FF} \vee \varphi = \mathit{FF})$
$\varphi \preceq \mathit{FNot}\ (\mathit{FVar}\ x) \longleftrightarrow (\varphi = \mathit{FNot}\ (\mathit{FVar}\ x) \vee \varphi = \mathit{FVar}\ x)$
⟨*proof*⟩

fun *simple-not-symb* **where**
simple-not-symb (*FNot* φ) = (*simple* φ) |
simple-not-symb - = *True*

definition *simple-not* **where**
simple-not = *all-subformula-st simple-not-symb*
declare *simple-not-def*[*simp*]

lemma *simple-not-Not*[*simp*]:
¬ *simple-not* (*FNot* (*FAnd* $\varphi\ \psi$))
¬ *simple-not* (*FNot* (*FOr* $\varphi\ \psi$))
⟨*proof*⟩

lemma *simple-not-step-exists*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ
shows $\psi \preceq \varphi \Longrightarrow \neg\ \mathit{simple\text{-}not\text{-}symb}\ \psi \Longrightarrow \exists \psi'.\ \mathit{pushNeg}\ \psi\ \psi'$
⟨*proof*⟩

lemma *simple-not-rew*:
fixes φ :: *'v propo*
assumes *noTB*: ¬ *simple-not* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge \mathit{pushNeg}\ \psi\ \psi'$
⟨*proof*⟩

lemma *no-T-F-except-top-level-pushNeg1*:

no-T-F-except-top-level (*FNot* (*FAnd* φ ψ)) $\Longrightarrow$ *no-T-F-except-top-level* (*FOr* (*FNot* φ) (*FNot* ψ))
⟨*proof*⟩

lemma *no-T-F-except-top-level-pushNeg2*:
no-T-F-except-top-level (*FNot* (*FOr* φ ψ)) $\Longrightarrow$ *no-T-F-except-top-level* (*FAnd* (*FNot* φ) (*FNot* ψ))
⟨*proof*⟩

lemma *no-T-F-symb-pushNeg*:
no-T-F-symb (*FOr* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FAnd* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FNot* (*FNot* φ'))
⟨*proof*⟩

lemma *propo-rew-step-pushNeg-no-T-F-symb*:
propo-rew-step pushNeg φ ψ $\Longrightarrow$ *no-T-F-except-top-level* φ $\Longrightarrow$ *no-T-F-symb* φ $\Longrightarrow$ *no-T-F-symb* ψ
⟨*proof*⟩

lemma *propo-rew-step-pushNeg-no-T-F*:
propo-rew-step pushNeg φ ψ $\Longrightarrow$ *no-T-F* φ $\Longrightarrow$ *no-T-F* ψ
⟨*proof*⟩

lemma *pushNeg-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step pushNeg*) φ ψ
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ
⟨*proof*⟩

lemma *pushNeg-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step pushNeg*) φ ψ **and**
no-T-F-except-top-level φ
shows *simple-not* ψ
⟨*proof*⟩

0.3.5 Push Inside

inductive *push-conn-inside* :: *'v connective* ⇒ *'v connective* ⇒ *'v propo* ⇒ *'v propo* ⇒ *bool*
for *c c'*:: *'v connective* **where**
push-conn-inside-l[*simp*]: $c = CAnd \lor c = COr \Longrightarrow c' = CAnd \lor c' = COr$
$\Longrightarrow$ *push-conn-inside c c'* (*conn c* [*conn c'* [$\varphi 1$, $\varphi 2$], ψ])
(*conn c'* [*conn c* [$\varphi 1$, ψ], *conn c* [$\varphi 2$, ψ]]) |
push-conn-inside-r[*simp*]: $c = CAnd \lor c = COr \Longrightarrow c' = CAnd \lor c' = COr$
$\Longrightarrow$ *push-conn-inside c c'* (*conn c* [ψ, *conn c'* [$\varphi 1$, $\varphi 2$]])
(*conn c'* [*conn c* [ψ, $\varphi 1$], *conn c* [ψ, $\varphi 2$]])

lemma *push-conn-inside-explicit*: *push-conn-inside c c'* φ ψ $\Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
⟨*proof*⟩

lemma *push-conn-inside-consistent*: *preserve-models* (*push-conn-inside c c'*)

⟨*proof*⟩

lemma *propo-rew-step-push-conn-inside*[*simp*]:
$\neg$*propo-rew-step* (*push-conn-inside c c′*) *FT* ψ $\neg$*propo-rew-step* (*push-conn-inside c c′*) *FF* ψ
⟨*proof*⟩

inductive *not-c-in-c′-symb*:: *′v connective* $\Rightarrow$ *′v connective* $\Rightarrow$ *′v propo* $\Rightarrow$ *bool* **for** *c c′* **where**
not-c-in-c′-symb-l[*simp*]: *wf-conn c* [*conn c′* [φ, φ'], ψ] $\Longrightarrow$ *wf-conn c′* [φ, φ']
$\Longrightarrow$ *not-c-in-c′-symb c c′* (*conn c* [*conn c′* [φ, φ'], ψ]) |
not-c-in-c′-symb-r[*simp*]: *wf-conn c* [ψ, *conn c′* [φ, φ']] $\Longrightarrow$ *wf-conn c′* [φ, φ']
$\Longrightarrow$ *not-c-in-c′-symb c c′* (*conn c* [ψ, *conn c′* [φ, φ']])

abbreviation *c-in-c′-symb c c′* φ $\equiv$ $\neg$*not-c-in-c′-symb c c′* φ

lemma *c-in-c′-symb-simp*:
not-c-in-c′-symb c c′ ξ $\Longrightarrow$ ξ = *FF* $\vee$ ξ = *FT* $\vee$ ξ = *FVar x* $\vee$ ξ = *FNot FF* $\vee$ ξ = *FNot FT*
$\vee$ ξ = *FNot* (*FVar x*)$\Longrightarrow$ *False*
⟨*proof*⟩

lemma *c-in-c′-symb-simp′*[*simp*]:
$\neg$*not-c-in-c′-symb c c′ FF*
$\neg$*not-c-in-c′-symb c c′ FT*
$\neg$*not-c-in-c′-symb c c′* (*FVar x*)
$\neg$*not-c-in-c′-symb c c′* (*FNot FF*)
$\neg$*not-c-in-c′-symb c c′* (*FNot FT*)
$\neg$*not-c-in-c′-symb c c′* (*FNot* (*FVar x*))
⟨*proof*⟩

definition *c-in-c′-only* **where**
c-in-c′-only c c′ $\equiv$ *all-subformula-st* (*c-in-c′-symb c c′*)

lemma *c-in-c′-only-simp*[*simp*]:
c-in-c′-only c c′ FF
c-in-c′-only c c′ FT
c-in-c′-only c c′ (*FVar x*)
c-in-c′-only c c′ (*FNot FF*)
c-in-c′-only c c′ (*FNot FT*)
c-in-c′-only c c′ (*FNot* (*FVar x*))
⟨*proof*⟩

lemma *not-c-in-c′-symb-commute*:
not-c-in-c′-symb c c′ ξ $\Longrightarrow$ *wf-conn c* [φ, ψ] $\Longrightarrow$ ξ = *conn c* [φ, ψ]
$\Longrightarrow$ *not-c-in-c′-symb c c′* (*conn c* [ψ, φ])
⟨*proof*⟩

lemma *not-c-in-c′-symb-commute′*:
wf-conn c [φ, ψ] $\Longrightarrow$ *c-in-c′-symb c c′* (*conn c* [φ, ψ]) $\longleftrightarrow$ *c-in-c′-symb c c′* (*conn c* [ψ, φ])
⟨*proof*⟩

lemma *not-c-in-c′-comm*:
assumes *wf*: *wf-conn c* [φ, ψ]
shows *c-in-c′-only c c′* (*conn c* [φ, ψ]) $\longleftrightarrow$ *c-in-c′-only c c′* (*conn c* [ψ, φ]) (**is** *?A* $\longleftrightarrow$ *?B*)
⟨*proof*⟩

lemma *not-c-in-c′-simp*[*simp*]:
fixes $\varphi 1$ $\varphi 2$ ψ :: *′v propo* **and** x :: *′v*
shows
c-in-c′-symb c c′ FT
c-in-c′-symb c c′ FF
c-in-c′-symb c c′ (*FVar x*)
wf-conn c [*conn c′* [$\varphi 1$, $\varphi 2$], ψ] $\Longrightarrow$ *wf-conn c′* [$\varphi 1$, $\varphi 2$]
$\Longrightarrow \neg$ *c-in-c′-only c c′* (*conn c* [*conn c′* [$\varphi 1$, $\varphi 2$], ψ])
⟨*proof*⟩

lemma *c-in-c′-symb-not*[*simp*]:
fixes *c c′* :: *′v connective* **and** ψ :: *′v propo*
shows *c-in-c′-symb c c′* (*FNot* ψ)
⟨*proof*⟩

lemma *c-in-c′-symb-step-exists*:
fixes φ :: *′v propo*
assumes *c*: $c = CAnd \lor c = COr$ **and** *c′*: $c' = CAnd \lor c' = COr$
shows $\psi \preceq \varphi \Longrightarrow \neg$ *c-in-c′-symb c c′* $\psi \Longrightarrow \exists \psi'$. *push-conn-inside c c′* ψ ψ'
⟨*proof*⟩

lemma *c-in-c′-symb-rew*:
fixes φ :: *′v propo*
assumes *noTB*: ¬*c-in-c′-only c c′* φ
and *c*: $c = CAnd \lor c = COr$ **and** *c′*: $c' = CAnd \lor c' = COr$
shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \land$ *push-conn-inside c c′* ψ ψ'
⟨*proof*⟩

lemma *push-conn-insidec-in-c′-symb-no-T-F*:
fixes φ ψ :: *′v propo*
shows *propo-rew-step* (*push-conn-inside c c′*) φ ψ $\Longrightarrow$ *no-T-F* φ $\Longrightarrow$ *no-T-F* ψ
⟨*proof*⟩

lemma *simple-propo-rew-step-push-conn-inside-inv*:
propo-rew-step (*push-conn-inside c c′*) φ ψ $\Longrightarrow$ *simple* φ $\Longrightarrow$ *simple* ψ
⟨*proof*⟩

lemma *simple-propo-rew-step-inv-push-conn-inside-simple-not*:
fixes *c c′* :: *′v connective* **and** φ ψ :: *′v propo*
shows *propo-rew-step* (*push-conn-inside c c′*) φ ψ $\Longrightarrow$ *simple-not* φ $\Longrightarrow$ *simple-not* ψ
⟨*proof*⟩

lemma *propo-rew-step-push-conn-inside-simple-not*:
fixes φ φ' :: *′v propo* **and** ξ ξ' :: *′v propo list* **and** *c* :: *′v connective*
assumes
propo-rew-step (*push-conn-inside c c′*) φ φ' **and**
wf-conn c (ξ @ φ # ξ') **and**
simple-not-symb (*conn c* (ξ @ φ # ξ')) **and**
simple-not-symb φ'
shows *simple-not-symb* (*conn c* (ξ @ φ' # ξ'))
⟨*proof*⟩

lemma *push-conn-inside-not-true-false*:
push-conn-inside c c' $\varphi\ \psi \Longrightarrow \psi \neq FT \wedge \psi \neq FF$
⟨*proof*⟩

lemma *push-conn-inside-inv*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *full* (*propo-rew-step* (*push-conn-inside c c'*)) $\varphi\ \psi$
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
⟨*proof*⟩

lemma *push-conn-inside-full-propo-rew-step*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* (*push-conn-inside c c'*)) $\varphi\ \psi$ **and**
no-T-F-except-top-level φ **and**
simple-not φ **and**
$c = CAnd \vee c = COr$ **and**
$c' = CAnd \vee c' = COr$
shows *c-in-c'-only c c'* ψ
⟨*proof*⟩

Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* :: *'v connective* ⇒ *'v propo* ⇒ *bool* **for** *c* :: *'v connective* **where**
simple-only-c-inside[*simp*]: *simple* $\varphi \Longrightarrow$ *only-c-inside-symb c* φ |
simple-cnot-only-c-inside[*simp*]: *simple* $\varphi \Longrightarrow$ *only-c-inside-symb c* (*FNot* φ) |
only-c-inside-into-only-c-inside: *wf-conn c l* $\Longrightarrow$ *only-c-inside-symb c* (*conn c l*)

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb c FF only-c-inside-symb c FT only-c-inside-symb c (*FVar x*) ⟨*proof*⟩

definition *only-c-inside* **where** *only-c-inside c* = *all-subformula-st* (*only-c-inside-symb c*)

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb c $\psi \longleftrightarrow$ (*simple* ψ
$\vee\ (\exists\ \varphi'.\ \psi = FNot\ \varphi' \wedge simple\ \varphi')$
$\vee\ (\exists\, l.\ \psi = conn\ c\ l \wedge wf\text{-}conn\ c\ l))$
⟨*proof*⟩

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes *c* :: *'v connective*
assumes *c*: $c \neq CNot$
shows *only-c-inside-symb c* (*FNot* ψ) $\longleftrightarrow$ *simple* ψ
⟨*proof*⟩

lemma *only-c-inside-decomp-not*[*simp*]:
assumes *c*: $c \neq CNot$
shows *only-c-inside c* (*FNot* ψ) $\longleftrightarrow$ *simple* ψ
⟨*proof*⟩

lemma *only-c-inside-decomp*:
only-c-inside c $\varphi \longleftrightarrow$
$(\forall \psi.\ \psi \preceq \varphi \longrightarrow (\mathit{simple}\ \psi \lor (\exists\ \varphi'.\ \psi = \mathit{FNot}\ \varphi' \land \mathit{simple}\ \varphi')$
$\lor (\exists\, l.\ \psi = \mathit{conn}\ c\ l \land \mathit{wf\text{-}conn}\ c\ l)))$
⟨*proof*⟩

lemma *only-c-inside-c-c′-false*:
fixes $c\ c' :: 'v$ *connective* **and** $l :: 'v$ *propo list* **and** $\varphi :: 'v$ *propo*
assumes cc': $c \neq c'$ **and** c: $c = \mathit{CAnd} \lor c = \mathit{COr}$ **and** c': $c' = \mathit{CAnd} \lor c' = \mathit{COr}$
and *only*: *only-c-inside c* φ **and** *incl*: $\mathit{conn}\ c'\ l \preceq \varphi$ **and** *wf*: *wf-conn* $c'\ l$
shows *False*
⟨*proof*⟩

lemma *only-c-inside-implies-c-in-c′-symb*:
assumes δ: $c \neq c'$ **and** c: $c = \mathit{CAnd} \lor c = \mathit{COr}$ **and** c': $c' = \mathit{CAnd} \lor c' = \mathit{COr}$
shows *only-c-inside c* $\varphi \Longrightarrow$ *c-in-c′-symb* $c\ c'\ \varphi$
⟨*proof*⟩

lemma *c-in-c′-symb-decomp-level1*:
fixes $l :: 'v$ *propo list* **and** $c\ c'\ ca :: 'v$ *connective*
shows *wf-conn ca l* $\Longrightarrow ca \neq c \Longrightarrow$ *c-in-c′-symb* $c\ c'$ (*conn ca l*)
⟨*proof*⟩

lemma *only-c-inside-implies-c-in-c′-only*:
assumes δ: $c \neq c'$ **and** c: $c = \mathit{CAnd} \lor c = \mathit{COr}$ **and** c': $c' = \mathit{CAnd} \lor c' = \mathit{COr}$
shows *only-c-inside c* $\varphi \Longrightarrow$ *c-in-c′-only* $c\ c'\ \varphi$
⟨*proof*⟩

lemma *c-in-c′-symb-c-implies-only-c-inside*:
assumes δ: $c = \mathit{CAnd} \lor c = \mathit{COr}\ c' = \mathit{CAnd} \lor c' = \mathit{COr}\ c \neq c'$ **and** *wf*: *wf-conn c* $[\varphi, \psi]$
and *inv*: *no-equiv* (*conn c l*) *no-imp* (*conn c l*) *simple-not* (*conn c l*)
shows *wf-conn c l* $\Longrightarrow$ *c-in-c′-only* $c\ c'$ (*conn c l*) $\Longrightarrow (\forall \psi \in \mathit{set}\ l.\ \mathit{only\text{-}c\text{-}inside}\ c\ \psi)$
⟨*proof*⟩

Push Conjunction

definition *pushConj* **where** *pushConj* = *push-conn-inside CAnd COr*

lemma *pushConj-consistent*: *preserve-models pushConj*
⟨*proof*⟩

definition *and-in-or-symb* **where** *and-in-or-symb* = *c-in-c′-symb CAnd COr*

definition *and-in-or-only* **where**
and-in-or-only = *all-subformula-st* (*c-in-c′-symb CAnd COr*)

lemma *pushConj-inv*:
fixes $\varphi\ \psi :: 'v$ *propo*
assumes *full* (*propo-rew-step pushConj*) $\varphi\ \psi$
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
⟨*proof*⟩

lemma *pushConj-full-propo-rew-step*:
  fixes φ ψ :: *'v propo*
  assumes
    no-equiv φ **and**
    no-imp φ **and**
    full (*propo-rew-step pushConj*) φ ψ **and**
    no-T-F-except-top-level φ **and**
    simple-not φ
  shows *and-in-or-only* ψ
  ⟨*proof*⟩

Push Disjunction

definition *pushDisj* **where** *pushDisj* = *push-conn-inside COr CAnd*

lemma *pushDisj-consistent*: *preserve-models pushDisj*
  ⟨*proof*⟩

definition *or-in-and-symb* **where** *or-in-and-symb* = *c-in-c'-symb COr CAnd*

definition *or-in-and-only* **where**
or-in-and-only = *all-subformula-st* (*c-in-c'-symb COr CAnd*)

lemma *not-or-in-and-only-or-and*[*simp*]:
  ~*or-in-and-only* (*FOr* (*FAnd* $\psi 1$ $\psi 2$) φ')
  ⟨*proof*⟩

lemma *pushDisj-inv*:
  fixes φ ψ :: *'v propo*
  assumes *full* (*propo-rew-step pushDisj*) φ ψ
  and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
  shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
  ⟨*proof*⟩

lemma *pushDisj-full-propo-rew-step*:
  fixes φ ψ :: *'v propo*
  assumes
    no-equiv φ **and**
    no-imp φ **and**
    full (*propo-rew-step pushDisj*) φ ψ **and**
    no-T-F-except-top-level φ **and**
    simple-not φ
  shows *or-in-and-only* ψ
  ⟨*proof*⟩

0.4 The Full Transformations

0.4.1 Abstract Definition

The normal form is a super group of groups

inductive *grouped-by* :: *'a connective* ⇒ *'a propo* ⇒ *bool* **for** *c* **where**
simple-is-grouped[*simp*]: *simple* $\varphi \Longrightarrow$ *grouped-by c* φ |
simple-not-is-grouped[*simp*]: *simple* $\varphi \Longrightarrow$ *grouped-by c* (*FNot* φ) |

connected-is-group[*simp*]: *grouped-by c* $\varphi \Longrightarrow$ *grouped-by c* $\psi \Longrightarrow$ *wf-conn c* $[\varphi, \psi]$
$\Longrightarrow$ *grouped-by c* (*conn c* $[\varphi, \psi]$)

lemma *simple-clause*[*simp*]:
grouped-by c FT
grouped-by c FF
grouped-by c (*FVar x*)
grouped-by c (*FNot FT*)
grouped-by c (*FNot FF*)
grouped-by c (*FNot* (*FVar x*))
⟨*proof*⟩

lemma *only-c-inside-symb-c-eq-c'*:
only-c-inside-symb c (*conn c'* $[\varphi 1, \varphi 2]$) $\Longrightarrow c' =$ *CAnd* $\vee\ c' =$ *COr* $\Longrightarrow$ *wf-conn c'* $[\varphi 1, \varphi 2]$
$\Longrightarrow c' = c$
⟨*proof*⟩

lemma *only-c-inside-c-eq-c'*:
only-c-inside c (*conn c'* $[\varphi 1, \varphi 2]$) $\Longrightarrow c' =$ *CAnd* $\vee\ c' =$ *COr* $\Longrightarrow$ *wf-conn c'* $[\varphi 1, \varphi 2] \Longrightarrow c = c'$
⟨*proof*⟩

lemma *only-c-inside-imp-grouped-by*:
assumes *c*: $c \neq$ *CNot* **and** *c'*: $c' =$ *CAnd* $\vee\ c' =$ *COr*
shows *only-c-inside c* $\varphi \Longrightarrow$ *grouped-by c* φ (**is** *?O* $\varphi \Longrightarrow$ *?G* φ)
⟨*proof*⟩

lemma *grouped-by-false*:
grouped-by c (*conn c'* $[\varphi, \psi]$) $\Longrightarrow c \neq c' \Longrightarrow$ *wf-conn c'* $[\varphi, \psi] \Longrightarrow$ *False*
⟨*proof*⟩

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

inductive *super-grouped-by*:: *'a connective* $\Rightarrow$ *'a connective* $\Rightarrow$ *'a propo* $\Rightarrow$ *bool* **for** *c c'* **where**
grouped-is-super-grouped[*simp*]: *grouped-by c* $\varphi \Longrightarrow$ *super-grouped-by c c'* φ |
connected-is-super-group: *super-grouped-by c c'* $\varphi \Longrightarrow$ *super-grouped-by c c'* $\psi \Longrightarrow$ *wf-conn c* $[\varphi, \psi]$
$\Longrightarrow$ *super-grouped-by c c'* (*conn c'* $[\varphi, \psi]$)

lemma *simple-cnf*[*simp*]:
super-grouped-by c c' FT
super-grouped-by c c' FF
super-grouped-by c c' (*FVar x*)
super-grouped-by c c' (*FNot FT*)
super-grouped-by c c' (*FNot FF*)
super-grouped-by c c' (*FNot* (*FVar x*))
⟨*proof*⟩

lemma *c-in-c'-only-super-grouped-by*:
assumes *c*: $c =$ *CAnd* $\vee\ c =$ *COr* **and** *c'*: $c' =$ *CAnd* $\vee\ c' =$ *COr* **and** *cc'*: $c \neq c'$
shows *no-equiv* $\varphi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *simple-not* $\varphi \Longrightarrow$ *c-in-c'-only c c'* φ
$\Longrightarrow$ *super-grouped-by c c'* φ
(**is** *?NE* $\varphi \Longrightarrow$ *?NI* $\varphi \Longrightarrow$ *?SN* $\varphi \Longrightarrow$ *?C* $\varphi \Longrightarrow$ *?S* φ)
⟨*proof*⟩

0.4.2 Conjunctive Normal Form

Definition

definition *is-conj-with-TF* **where** *is-conj-with-TF* == *super-grouped-by COr CAnd*

lemma *or-in-and-only-conjunction-in-disj*:
shows *no-equiv* $\varphi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *simple-not* $\varphi \Longrightarrow$ *or-in-and-only* $\varphi \Longrightarrow$ *is-conj-with-TF* φ
⟨*proof*⟩

definition *is-cnf* **where**
is-cnf $\varphi \equiv$ *is-conj-with-TF* $\varphi \wedge$ *no-T-F-except-top-level* φ

Full CNF transformation

The full1 CNF transformation consists simply in chaining all the transformation defined before.

definition *cnf-rew* **where** *cnf-rew* =
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew-equivalent*: *preserve-models cnf-rew*
⟨*proof*⟩

lemma *cnf-rew-is-cnf*: *cnf-rew* φ $\varphi' \Longrightarrow$ *is-cnf* φ'
⟨*proof*⟩

0.4.3 Disjunctive Normal Form

Definition

definition *is-disj-with-TF* **where** *is-disj-with-TF* $\equiv$ *super-grouped-by CAnd COr*

lemma *and-in-or-only-conjunction-in-disj*:
shows *no-equiv* $\varphi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *simple-not* $\varphi \Longrightarrow$ *and-in-or-only* $\varphi \Longrightarrow$ *is-disj-with-TF* φ
⟨*proof*⟩

definition *is-dnf* :: *'a propo* ⇒ *bool* **where**
is-dnf $\varphi \longleftrightarrow$ *is-disj-with-TF* $\varphi \wedge$ *no-T-F-except-top-level* φ

Full DNF transform

The full1 DNF transformation consists simply in chaining all the transformation defined before.

definition *dnf-rew* **where** *dnf-rew* $\equiv$
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushConj*))

lemma *dnf-rew-consistent*: *preserve-models dnf-rew*
⟨*proof*⟩

theorem *dnf-transformation-correction*:
  dnf-rew φ φ' $\Longrightarrow$ *is-dnf* φ'
⟨*proof*⟩

0.5 More aggressive simplifications: Removing true and false at the beginning

0.5.1 Transformation

We should remove *FT* and *FF* at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive *elimTBFull* **where**
ElimTBFull1[*simp*]: *elimTBFull* (*FAnd* φ *FT*) φ |
ElimTBFull1'[*simp*]: *elimTBFull* (*FAnd* *FT* φ) φ |

ElimTBFull2[*simp*]: *elimTBFull* (*FAnd* φ *FF*) *FF* |
ElimTBFull2'[*simp*]: *elimTBFull* (*FAnd* *FF* φ) *FF* |

ElimTBFull3[*simp*]: *elimTBFull* (*FOr* φ *FT*) *FT* |
ElimTBFull3'[*simp*]: *elimTBFull* (*FOr* *FT* φ) *FT* |

ElimTBFull4[*simp*]: *elimTBFull* (*FOr* φ *FF*) φ |
ElimTBFull4'[*simp*]: *elimTBFull* (*FOr* *FF* φ) φ |

ElimTBFull5[*simp*]: *elimTBFull* (*FNot* *FT*) *FF* |
ElimTBFull5'[*simp*]: *elimTBFull* (*FNot* *FF*) *FT* |

ElimTBFull6-l[*simp*]: *elimTBFull* (*FImp* *FT* φ) φ |
ElimTBFull6-l'[*simp*]: *elimTBFull* (*FImp* *FF* φ) *FT* |
ElimTBFull6-r[*simp*]: *elimTBFull* (*FImp* φ *FT*) *FT* |
ElimTBFull6-r'[*simp*]: *elimTBFull* (*FImp* φ *FF*) (*FNot* φ) |

ElimTBFull7-l[*simp*]: *elimTBFull* (*FEq* *FT* φ) φ |
ElimTBFull7-l'[*simp*]: *elimTBFull* (*FEq* *FF* φ) (*FNot* φ) |
ElimTBFull7-r[*simp*]: *elimTBFull* (*FEq* φ *FT*) φ |
ElimTBFull7-r'[*simp*]: *elimTBFull* (*FEq* φ *FF*) (*FNot* φ)

The transformation is still consistent.

lemma *elimTBFull-consistent*: *preserve-models elimTBFull*
⟨*proof*⟩

Contrary to the theorem *no-T-F-symb-except-toplevel-step-exists*, we do not need the assumption *no-equiv* φ and *no-imp* φ, since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:
  fixes φ :: *'v propo*
  shows $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists\,\psi'$. *elimTBFull* ψ ψ'
⟨*proof*⟩

The same applies here. We do not need the assumption, but the deep link between $\neg$ *no-T-F-except-top-level* φ and the existence of a rewriting step, still exists.

lemma *no-T-F-except-top-level-rew'*:
  fixes φ :: *'v propo*
  assumes *noTB*: $\neg$ *no-T-F-except-top-level* φ

shows $\exists\,\psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elimTBFull* $\psi\ \psi'$
⟨*proof*⟩

lemma *elimTBFull-full-propo-rew-step*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *full* (*propo-rew-step elimTBFull*) $\varphi\ \psi$
shows *no-T-F-except-top-level* ψ
⟨*proof*⟩

0.5.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

lemma *propo-rew-step-ElimEquiv-no-T-F*: *propo-rew-step elim-equiv* $\varphi\ \psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
⟨*proof*⟩

lemma *elim-equiv-inv'*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *full* (*propo-rew-step elim-equiv*) $\varphi\ \psi$ **and** *no-T-F-except-top-level* φ
shows *no-T-F-except-top-level* ψ
⟨*proof*⟩

lemma *propo-rew-step-ElimImp-no-T-F*: *propo-rew-step elim-imp* $\varphi\ \psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
⟨*proof*⟩

lemma *elim-imp-inv'*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *full* (*propo-rew-step elim-imp*) $\varphi\ \psi$ **and** *no-T-F-except-top-level* φ
shows*no-T-F-except-top-level* ψ
⟨*proof*⟩

0.5.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

definition *dnf-rew'* :: *'a propo* ⇒ *'a propo* ⇒ *bool* **where**
dnf-rew' =
(*full* (*propo-rew-step elimTBFull*)) *OO*
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushConj*))

lemma *dnf-rew'-consistent*: *preserve-models dnf-rew'*
⟨*proof*⟩

theorem *cnf-transformation-correction*:
dnf-rew' $\varphi\ \varphi' \Longrightarrow$ *is-dnf* φ'
⟨*proof*⟩

Given all the lemmas before the CNF transformation is easy to prove:

definition *cnf-rew′* :: *′a propo* ⇒ *′a propo* ⇒ *bool* **where**
cnf-rew′ =
(*full* (*propo-rew-step elimTBFull*)) *OO*
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew′-consistent*: *preserve-models cnf-rew′*
⟨*proof*⟩

theorem *cnf′-transformation-correction*:
cnf-rew′ φ φ' ⟹ *is-cnf* φ'
⟨*proof*⟩

end
theory *Prop-Logic-Multiset*
imports *Nested-Multisets-Ordinals.Multiset-More Prop-Normalisation*
Entailment-Definition.Partial-Herbrand-Interpretation
begin

0.6 Link with Multiset Version

0.6.1 Transformation to Multiset

fun *mset-of-conj* :: *′a propo* ⇒ *′a literal multiset* **where**
mset-of-conj (*FOr* φ ψ) = *mset-of-conj* φ + *mset-of-conj* ψ |
mset-of-conj (*FVar v*) = {# *Pos v* #} |
mset-of-conj (*FNot* (*FVar v*)) = {# *Neg v* #} |
mset-of-conj FF = {#}

fun *mset-of-formula* :: *′a propo* ⇒ *′a literal multiset set* **where**
mset-of-formula (*FAnd* φ ψ) = *mset-of-formula* φ ∪ *mset-of-formula* ψ |
mset-of-formula (*FOr* φ ψ) = {*mset-of-conj* (*FOr* φ ψ)} |
mset-of-formula (*FVar* ψ) = {*mset-of-conj* (*FVar* ψ)} |
mset-of-formula (*FNot* ψ) = {*mset-of-conj* (*FNot* ψ)} |
mset-of-formula FF = {{#}} |
mset-of-formula FT = {}

0.6.2 Equisatisfiability of the two Versions

lemma *is-conj-with-TF-FNot*:
is-conj-with-TF (*FNot* φ) ⟷ (∃ *v*. φ = *FVar v* ∨ φ = *FF* ∨ φ = *FT*)
⟨*proof*⟩

lemma *grouped-by-COr-FNot*:
grouped-by COr (*FNot* φ) ⟷ (∃ *v*. φ = *FVar v* ∨ φ = *FF* ∨ φ = *FT*)
⟨*proof*⟩

lemma
shows *no-T-F-FF*[*simp*]: ¬*no-T-F FF* **and**
no-T-F-FT[*simp*]: ¬*no-T-F FT*
⟨*proof*⟩

lemma *grouped-by-CAnd-FAnd*:
grouped-by CAnd (*FAnd* $\varphi 1$ $\varphi 2$) ⟷ *grouped-by CAnd* $\varphi 1$ ∧ *grouped-by CAnd* $\varphi 2$

⟨*proof*⟩

lemma *grouped-by-COr-FOr*:
  $grouped\text{-}by\ COr\ (FOr\ \varphi 1\ \varphi 2) \longleftrightarrow grouped\text{-}by\ COr\ \varphi 1 \wedge grouped\text{-}by\ COr\ \varphi 2$
  ⟨*proof*⟩

lemma *grouped-by-COr-FAnd*[*simp*]: $\neg\ grouped\text{-}by\ COr\ (FAnd\ \varphi 1\ \varphi 2)$
  ⟨*proof*⟩

lemma *grouped-by-COr-FEq*[*simp*]: $\neg\ grouped\text{-}by\ COr\ (FEq\ \varphi 1\ \varphi 2)$
  ⟨*proof*⟩

lemma [*simp*]: $\neg grouped\text{-}by\ COr\ (FImp\ \varphi\ \psi)$
  ⟨*proof*⟩

lemma [*simp*]: $\neg\ is\text{-}conj\text{-}with\text{-}TF\ (FImp\ \varphi\ \psi)$
  ⟨*proof*⟩

lemma [*simp*]: $\neg\ is\text{-}conj\text{-}with\text{-}TF\ (FEq\ \varphi\ \psi)$
  ⟨*proof*⟩

lemma *is-conj-with-TF-Fand*:
  $is\text{-}conj\text{-}with\text{-}TF\ (FAnd\ \varphi 1\ \varphi 2) \Longrightarrow is\text{-}conj\text{-}with\text{-}TF\ \varphi 1 \wedge\ is\text{-}conj\text{-}with\text{-}TF\ \varphi 2$
  ⟨*proof*⟩

lemma *is-conj-with-TF-FOr*:
  $is\text{-}conj\text{-}with\text{-}TF\ (FOr\ \varphi 1\ \varphi 2) \Longrightarrow grouped\text{-}by\ COr\ \varphi 1 \wedge grouped\text{-}by\ COr\ \varphi 2$
  ⟨*proof*⟩

lemma *grouped-by-COr-mset-of-formula*:
  $grouped\text{-}by\ COr\ \varphi \Longrightarrow mset\text{-}of\text{-}formula\ \varphi = (if\ \varphi = FT\ then\ \{\}\ else\ \{mset\text{-}of\text{-}conj\ \varphi\})$
  ⟨*proof*⟩

When a formula is in CNF form, then there is equisatisfiability between the multiset version and the CNF form. Remark that the definition for the entailment are slightly different: ($\models$) uses a function assigning *True* or *False*, while ($\models s$) uses a set where being in the list means entailment of a literal.

theorem *cnf-eval-true-clss*:
  fixes φ :: *'v propo*
  assumes *is-cnf* φ
  shows $eval\ A\ \varphi \longleftrightarrow Partial\text{-}Herbrand\text{-}Interpretation.true\text{-}clss\ (\{Pos\ v|v.\ A\ v\} \cup \{Neg\ v|v.\ \neg A\ v\})$
    $(mset\text{-}of\text{-}formula\ \varphi)$
  ⟨*proof*⟩

function *formula-of-mset* :: *'a clause* ⇒ *'a propo* **where**

```
‹formula-of-mset φ =
    (if φ = {#} then FF
     else
        let v = (SOME v. v ∈# φ);
            v' = (if is-pos v then FVar (atm-of v) else FNot (FVar (atm-of v))) in
        if remove1-mset v φ = {#} then v'
        else FOr v' (formula-of-mset (remove1-mset v φ)))›
```

  ⟨*proof*⟩
termination

⟨*proof*⟩

lemma *formula-of-mset-empty*[*simp*]: ‹*formula-of-mset* {#} = *FF*›
⟨*proof*⟩

lemma *formula-of-mset-empty-iff*[*iff*]: ‹*formula-of-mset* φ = *FF* ⟷ φ = {#}›
⟨*proof*⟩

declare *formula-of-mset.simps*[*simp del*]

function *formula-of-msets* :: *'a literal multiset set* ⇒ *'a propo* **where**
‹*formula-of-msets* φ*s* =
(*if* φ*s* = {} ∨ *infinite* φ*s then FT*
else
let v = (*SOME v. v* ∈ φ*s*);
v' = *formula-of-mset v in*
if φ*s* − {*v*} = {} *then v'*
else FAnd v' (*formula-of-msets* (φ*s* − {*v*})))›
⟨*proof*⟩
termination
⟨*proof*⟩

declare *formula-of-msets.simps*[*simp del*]

lemma *remove1-mset-empty-iff*:
‹*remove1-mset v* φ = {#} ⟷ (φ = {#} ∨ φ = {#*v*#})›
⟨*proof*⟩

definition *fun-of-set* **where**
‹*fun-of-set A x* = (*if Pos x* ∈ *A then True else if Neg x* ∈ *A then False else undefined*)›

lemma *grouped-by-COr-formula-of-mset*: ‹*grouped-by COr* (*formula-of-mset* φ)›
⟨*proof*⟩
lemma *no-T-F-formula-of-mset*: ‹*no-T-F* (*formula-of-mset* φ)› **if** ‹*formula-of-mset* φ ≠ *FF*› **for** φ
⟨*proof*⟩

lemma *mset-of-conj-formula-of-mset*[*simp*]: ‹*mset-of-conj*(*formula-of-mset* φ) = φ› **for** φ
⟨*proof*⟩

lemma *mset-of-formula-formula-of-mset* [*simp*]: ‹*mset-of-formula* (*formula-of-mset* φ) = {φ}› **for** φ
⟨*proof*⟩

lemma *formula-of-mset-is-cnf*: ‹*is-cnf* (*formula-of-mset* φ)›
⟨*proof*⟩

lemma *eval-clss-iff*:
assumes ‹*consistent-interp A*› **and** ‹*total-over-set A UNIV*›
shows ‹*eval* (*fun-of-set A*) (*formula-of-mset* φ) ⟷ *Partial-Herbrand-Interpretation.true-clss A* {φ}›
⟨*proof*⟩

lemma *is-conj-with-TF-Fand-iff*:
is-conj-with-TF (*FAnd* φ*1* φ*2*) ⟷ *is-conj-with-TF* φ*1* ∧ *is-conj-with-TF* φ*2*
⟨*proof*⟩

lemma *is-CNF-Fand*:
‹*is-cnf* (*FAnd* φ ψ) ⟷ (*is-cnf* φ ∧ *no-T-F* φ) ∧ *is-cnf* ψ ∧ *no-T-F* ψ›

⟨*proof*⟩

lemma *no-T-F-formula-of-mset-iff*: ‹*no-T-F* (*formula-of-mset* φ) ⟷ $\varphi \neq \{\#\}$›
⟨*proof*⟩

lemma *no-T-F-formula-of-msets*:
assumes ‹*finite* φ› **and** ‹$\{\#\} \notin \varphi$› **and** ‹$\varphi \neq \{\}$›
shows ‹*no-T-F* (*formula-of-msets* (φ))›
⟨*proof*⟩

lemma *is-cnf-formula-of-msets*:
assumes ‹*finite* φ› **and** ‹$\{\#\} \notin \varphi$›
shows ‹*is-cnf* (*formula-of-msets* φ)›
⟨*proof*⟩

lemma *mset-of-formula-formula-of-msets*:
assumes ‹*finite* φ›
shows ‹*mset-of-formula* (*formula-of-msets* φ) = φ›
⟨*proof*⟩

lemma
assumes ‹*consistent-interp A*› **and** ‹*total-over-set A UNIV*› **and** ‹*finite* φ› **and** ‹$\{\#\} \notin \varphi$›
shows ‹*eval* (*fun-of-set A*) (*formula-of-msets* φ) ⟷ *Partial-Herbrand-Interpretation.true-clss A* φ›
⟨*proof*⟩

end