A Note on the Complexity of Real Algebraic Hypersurfaces

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Abstract. Given an algebraic hypersurface $\mathcal{O}$ in $\mathbb{R}^d$, how many simplices are necessary for a simplicial complex isotopic to $\mathcal{O}$? We address this problem and the variant where all vertices of the complex must lie on $\mathcal{O}$. We give asymptotically tight worst-case bounds for algebraic plane curves of degree n. Our results gradually improve known bounds in higher dimensions; however, the question for tight bounds remains unsolved for $d \geq 3$.

1 Introduction

A standard technique to process non-linear curves and surfaces in geometric systems is to approximate them in terms of a piecewise linear object (a simplicial complex). A main goal is to preserve the topological properties of the input. Furthermore, geometric properties, such as the position of singular or "extremal" points of the object are often of interest. For algebraic curves and surfaces as inputs, the former problem is usually called *topology computation*, the latter *topological-geometric analysis* of the object.

We consider the following question: *How many simplices are needed to embed a simplicial complex in $\mathbb{R}^d$ that is isotopic to a real algebraic hypersurface*[1] *in $\mathbb{R}^d$ of degree n?* Our main contribution is to provide sharp bounds for the planar case ($d = 2$): for a topologically correct representation, $\Omega(n^2)$ line segments are needed in the worst case, and we give an algorithm producing $O(n^2)$ line segments for all cases. Although the idea is simple, it seemingly does not appear in the literature yet. For geometric-topological representations, we construct a class of curves such that $\Omega(n^3)$ line segments are necessary. This proves that the *cylindrical algebraic decomposition* [5] ("Find the critical x-coordinates of the curve; compute the fiber at these coordinates and at separating points in between; connect the fiber points by straight-line segments." – compare Fig. 2) is asymptotically optimal. This is surprising because the vertical decomposition strategy seems to introduce much more line segments than actually necessary.

Our results can be partially generalized in higher dimensions. This allows a gradual improvement of lower and upper bounds that can be derived easily from cylindrical algebraic decomposition. Nevertheless, our bounds fail to be tight already for algebraic surfaces: For the topological approximation, we get a lower bound of $\Omega(n^3)$, and an upper bound of $O(n^5)$ triangles. For the geometric-topological approximation, the bounds

[1] A real algebraic hypersurface $\mathcal{O} \subset \mathbb{R}^d$ is defined as the real vanishing set $\mathcal{O} = V(f) = \{x \in \mathbb{R}^d : f(x) = 0\}$ of a polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$. Its degree is given by the degree of f.

are $\Omega(n^4)$ and $O(n^7)$, respectively. These gaps increase in higher dimensions because the lower bounds grow single exponentially in the dimension, whereas the upper bounds grow double exponentially.

Related work: Efficient techniques for topology computation of algebraic curves (e.g, see [6, 9], and references therein) and surfaces [4, 1] have been presented in case where the defining polynomial f has integer coefficients. For the planar case, the complexity of the problem has been upper bounded by $O(N^{12})$ [8, 10], where N is defined as the maximum of the degree of f and the bitsize of its coefficients. However, our question of how many segment/triangles are needed in principal to capture the topology of the object seems to be untreated in this context.

We remark that similar problems have been extensively studied for 2-manifolds. For instance, Nakamoto and Ota [12] show that any closed compact 2-manifold of genus g can be triangulated using $\Theta(g)$ vertices. An often discussed concept in this context is an *irreducible triangulation* of a 2-manifold, i.e., a triangulation where no edge can be contracted without changing the topology. It has been shown that only finitely many irreducible triangulations exist [2], and they have been enumerated explicitly for the torus [11]. Although these results aim in a somewhat similar direction, algebraic surfaces are in general not 2-manifolds and need different techniques to be analyzed.

2 Basic notation and definitions

A *homeomorphism* between two sets $X, Y \subset \mathbb{R}^d$ is a bijective, continuous map $h : X \to Y$ whose inverse is continuous as well. X and Y are *isotopic* if they are "connected by homeomorphism", that is, there exists a continuous map $\psi : [0,1] \times X \to \mathbb{R}^d$ such that $\psi(0,x) = x$, $\psi(1,X) = Y$, and $\psi(t_0,x)$ is a homeomorphism for any $t_0 \in [0,1]$. ψ is called an *isotopy* between X and Y. We assume that the reader is familiar with the definition of a simplicial complex. We assume that the complex is embedded into $\mathbb{R}^d$ by fixing its vertices, and we identify the complex and the induced point set.

An *algebraic hypersurface* $\mathscr{C}$ in $\mathbb{R}^d$ is the solution set of an equation $f = 0$ with $f \in \mathbb{R}[x_1, \ldots, x_d]$. Hypersurfaces in dimensions 2 and 3 are called *algebraic curves* and *algebraic surfaces*, respectively. The *degree* of $\mathscr{C}$ is defined by the degree of f. An *isolated point* $p \in \mathbb{R}^d$ is a point on $\mathscr{C}$ such that an open neighborhood of p in $\mathbb{R}^d$ does not contain any further point of $\mathscr{C}$. An *isocomplex* of $\mathscr{C}$ is a simplicial complex S that is isotopic to $\mathscr{C}$. A *stable isocomplex* is an isocomplex that is stable at vertices, that is, there exists an isotopy ψ between $\mathscr{C}$ and S such that for each vertex v of S, $\psi(t_0, v) = v$ for any $t \in [0,1]$. Computing the topology of $\mathscr{C}$ means to compute an isocomplex, computing a geometric-topological analysis means to compute a stable isocomplex.

3 Bounds for algebraic plane curves

For simplicity, we assume throughout this section that any algebraic curve is bounded. Unbounded curves can be isotopically approximated within a sufficiently large bounding box using the same methodology.

3.1 Stable isocomplexes

Our main idea for deriving lower bounds is to construct algebraic hypersurfaces with many isolated points. We can even fix the location of each isolated point to a ball of arbitrary small radius.

Theorem 1. *For $d, n \in \mathbb{N}$, set $c := \binom{\lfloor n/2 \rfloor + d}{d} - d$. Then, for any $\varepsilon > 0$, and any set of points $p_1, \ldots, p_c \in \mathbb{Q}^d$, there exists a hypersurface $C \subset \mathbb{R}^d$ of degree n such that for any p_i, C contains an isolated point $p_i' \in \mathbb{R}^d$ with $\|p_i - p_i'\|_2 < \varepsilon$.*

Proof. W.l.o.g., we assume that n is even. The idea is to construct d polynomials $f_1, \ldots, f_d$ of degree $n/2$ that all interpolate the points $p_1, \ldots, p_c$, and to consider the curve defined by $f := f_1^2 + \ldots + f_d^2$. Obviously, $\deg f \leq n$, and $V(f)$ has isolated points exactly at the intersection points of $V(f_1) \cap \ldots \cap V(f_d)$. We have to prove that $f_1, \ldots, f_d$ can be chosen such that they intersect only in a finite number of points.

Firstly, almost all choices of d hypersurfaces of degree $n/2$ yield a zero-dimensional common intersection: consider the coefficients of the polynomials as indeterminates, then the (multivariate) resultant R [7] with respect to any variable, say x_1, is a polynomial in x_1 that does not vanish completely. Thus, for almost any choice of coefficients, the concrete set of polynomials will only have finitely many common intersections.

We next fix c points $p_1', \ldots, p_c'$ in $\mathbb{C}^d$ with yet indeterminate coordinates. We force d hypersurfaces with indeterminate coefficients to pass through them. As a consequence, each coefficient can be re-expressed in dependency of the coordinates of the p_i', plus additional degrees of freedom. The same also holds true for the resultant polynomial R. The statement of the theorem follows if we can prove that the resultant polynomial does not vanish identically for some choice of $p_1', \ldots, p_c'$, because this already implies that it does not vanish identically for almost all choices of $p_1', \ldots, p_c'$.

The degree of R is $(n/2)^d$. Choose d hypersurfaces $f_1, \ldots, f_d$ such that the leading term of R does not vanish. Then, there exist $(n/2)^d$ intersection points in the projective space $\mathbb{P}(\mathbb{C}^d)$, and we can w.l.o.g. assume that all these points actually lie in the affine space $\mathbb{C}^d$. It is a simple proof that $(n/2)^d \geq c$ for all $n, d \in \mathbb{N}$ (by induction on d). So, we can pick c of the common intersection points as points $p_1', \ldots, p_c'$ from above, and set the other degrees of freedom such that we obtain $f_1, \ldots, f_d$. With this choice, the resultant does not vanish, thus, it defines a lower-dimensional variety in $\mathbb{C}^d$. It follows that the resultant does not vanish for almost any choice of base points $p_1', \ldots, p_c'$.

Thus, for given points $p_1, \ldots, p_c \in \mathbb{Q}^d$, we find points $p_1', \ldots, p_c'$ in an ε-ball around them such that there are hypersurfaces $f_1, \ldots, f_d$ interpolating them and such that the resultant of $f_1, \ldots, f_d$ does not vanish completely. It remains to argue that $p_1', \ldots, p_c'$ can be chosen with real coordinates, but this follows immediately, since otherwise, the resultant variety would contain an open ball of $\mathbb{R}^d$, and consequently, it would contain the whole $\mathbb{R}^d$, which is impossible. □

For constant d, the theorem says that we can choose $\Theta(n^d)$ arbitrary rational points and construct an algebraic hypersurface of degree n with isolated points close to them.

Theorem 2. *There exists an algebraic curve $\mathscr{C} \subset \mathbb{R}^2$ of degree n such that any stable isocomplex for $\mathscr{C}$ has $\Omega(n^3)$ vertices.*

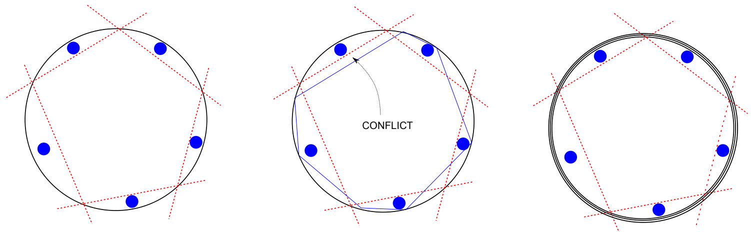

Fig. 1. Illustration of the construction in the proof of Theorem 2

Proof. We proof the claim by constructing a suitable curve $\mathscr{C}$. Let first be the unit circle be a component of $\mathscr{C}$. Any isocomplex of $\mathscr{C}$ must then contain a sequence of points on the unit circle which form a cycle in the complex. We cut out $c' := \binom{n/4+2}{2} - 2$ disjoint regions of the unit disc by intersecting the disc with c' different lines. We place a disc of size ε in each of the regions and force an isolated point of the curve $\mathscr{C}$ to lie inside each disc (Fig. 1 left). By Theorem 1, this is possible if $\mathscr{C}$ is of degree at least $n/2$.

The isotopic cycle for the unit circle component contains a vertex in each of the regions: If there is no such vertex, the cycle misses the region completely, so the isolated point is outside the cycle, contradicting the properties of a stable isocomplex (Fig. 1 middle). Hence, at least $c' = \Omega(n^2)$ vertices are placed on the unit circle.

Finally, we take a collection of $n/4$ concentric circles to be part of $\mathscr{C}$ (instead of just the unit circle) such that the lines chosen as above till cut out c' disjoint regions for any of the circles (Fig. 1 right). This is clearly possible, if all concentric circles have radius close enough to 1. The argument from above now works separately for each of the circles, thus, each one is divided into $\Omega(n^2)$ line segments under the isotopy.

To summarize, the final curve consists of two components: one curve of degree $n/2$ that forces the isolated singularities in the regions, and a collection of $n/4$ circles (of degree $n/2$). The union is of degree n, and any stable isocomplex requires $\Omega(n^3)$ line segments (and vertices) in total. □

The upper bound of $O(n^3)$ vertices is well-known and follows immediately from cylindrical algebraic decomposition (see Figure 2)

Proposition 1. *For any curve $\mathscr{C} \subset \mathbb{R}^2$ of degree n, there exists a stable isocomplex with $O(n^3)$ cells.*

3.2 General Isocomplexes

We next remove the stability requirement on the isocomplex. The following lower bound follows directly by considering an arrangement of n lines in generic position.

Proposition 2. *For any $n \in \mathbb{N}$, there exists an algebraic curve $\mathscr{C} \subset \mathbb{R}^2$ of degree n such that any isocomplex for $\mathscr{C}$ has $\Omega(n^2)$ vertices.*

In order to establish the upper bound of $O(n^2)$ for isocomplexes of algebraic curves, we show first that an algebraic curve decomposes into up to $O(n^2)$ points and smooth, x-monotone segments.

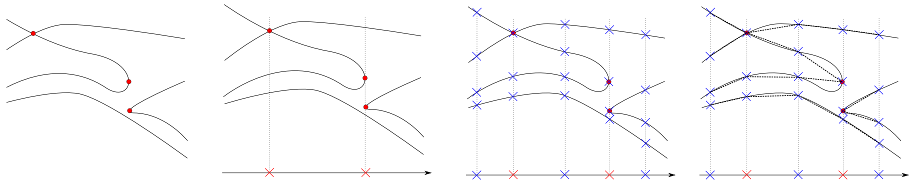

Fig. 2. An algebraic curve $\mathscr{C}$ of degree n has up to $n(n-1)$ x-critical points p, that is, $f(p) = f_y(p) = 0$. The projections of these points decompose the x-axes into $O(n^2)$ delineable sets. This means that the fiber above each cell in the decomposition consists of finitely many (at most n) function graphs. Inserting points in between two consecutive projections and lifting each of the points in one dimension leads to a stable isocomplex of $\mathscr{C}$ with at most $2n^2(n-1)$ points.

Definition 1. *Let $\mathscr{C} \subset \mathbb{R}^2$ be an algebraic curve without vertical segments. For a point $p \in \mathbb{R}^2$, the* branch numbers *of p are a pair of integers (ℓ_p, r_p) denoting the number of paths of the curves entering from the left hand side and from the right hand side, respectively. A point is called* event point *if its branch numbers do not equal $(1,1)$.*

Lemma 1. *For an event point p, we set b_p to the sum of its branch numbers. The sum of the b_p's for all event points is bounded by $2n(n-1)$.*

Proof. For a point $p = (x_0, y_0)$ on an algebraic plane curve $\mathscr{C} = V(f)$, we consider the Taylor expansion

$$f(x,y) = \sum_{i=0}^{n} (a_{i0}(x-x_0)^i + a_{i1}(x-x_0)^{i-1}(y-y_0) + \ldots + a_{ii}(y-y_0)^i)$$

of f at p. The smallest i such that at least one of the coefficients a_{ij}, $0 \leq j \leq i$, does not vanish is denoted the *multiplicity* $m_{\mathscr{C}}(p)$ of $\mathscr{C}$ at p. From this definition, it follows that $\mathscr{C}' := V(\frac{\partial f}{\partial y})$ has multiplicity $m_{\mathscr{C}'}(p) \geq m_{\mathscr{C}}(p) - 1$ at p. Furthermore, the *intersection multiplicity* $i(\mathscr{C}_1, \mathscr{C}_2, p)$ of two algebraic curves $\mathscr{C}_1 = V(f)$ and $\mathscr{C}_2 = V(g)$ at a point $p \in \mathbb{C}^2$ is defined as the dimension of the vector space $\mathbb{C}[x,y]_p/(f,g)$ where $\mathbb{C}[x,y]_p$ is the localization of the polynomial ring $\mathbb{C}[x,y]$ at p [3]. It holds that $m_{\mathscr{C}_1}(p) \cdot m_{\mathscr{C}_2} \leq i(\mathscr{C}_1, \mathscr{C}_2, p)$ with equality occurring iff f and g have no tangent line in common at p. Furthermore, due to Bézout's Theorem, the sum $\sum_{p \in \mathscr{C}_1 \cap \mathscr{C}_2} i(\mathscr{C}_1, \mathscr{C}_2, p)$ of all intersection multiplicities is bounded by $\deg(f) \cdot \deg(g)$.

If $p = (x_0, y_0)$ is no common intersection point of $\mathscr{C} := V(f)$ and $\mathscr{C}' := V(\frac{\partial f}{\partial y})$, then p is adjacent to exactly two arcs of $\mathscr{C}$ which are orthogonal to the gradient $\nabla f(p) = \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right)$ at p. Thus, the branch numbers for p are $(1,1)$. An event point $p = (x_0, y_0)$ is a common intersection point of $\mathscr{C}$ and $\mathscr{C}' := V(\frac{\partial f}{\partial y})$ and, hence, $i(\mathscr{C}, \mathscr{C}', p) \geq 1$ for each event point. The arithmetic mean $(\ell_p + r_p)/2$ of the two branch numbers ℓ_p and r_p at p constitutes a lower bound on the multiplicity of $\mathscr{C}$ at p; this follows from the fact that, for arbitrary small ε, there exists lines $L_x = V(x - x_0 + \varepsilon_x)$ and $L_y = V(y - y_0 + \varepsilon_y)$, $|\varepsilon_x|, |\varepsilon_y| < \varepsilon$, that both intersect $\mathscr{C}$ in at least $(\ell_p + r_p)/2$ points. This shows that the first $\lceil (\ell_p + r_p)/2 \rceil$ –order terms of the Taylor expansion of f at p vanish.

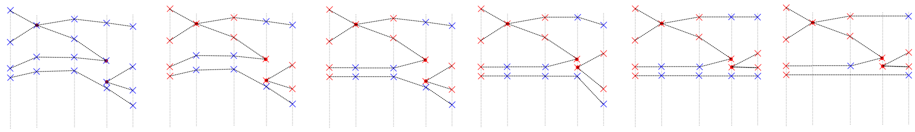

Fig. 3. Starting with the stable isocomplex of size $O(n^3)$ we straighten edges which connect two non critical points. Finally adjacent straight line connections are removed. The size of so obtained isocomplex reduces to the number of arcs of $\mathscr{C}$ connecting two critical points, that is, $O(n^2)$.

It follows that

$$\sum_{p \text{ event point}} (\ell_p + r_p) \leq 2 \cdot \sum_{p \text{ event point}} m_{\mathscr{C}}(p) \leq 2 \cdot \sum_{p \text{ event point}} m_{\mathscr{C}}(p) \cdot m_{\mathscr{C}'}(p)$$
$$\leq 2 \cdot \sum_{p \in \mathscr{C}} i(\mathscr{C}, \mathscr{C}', p) \leq 2n(n-1) \qquad \square$$

Theorem 3. *For any algebraic curve $\mathscr{C} \subset \mathbb{R}^2$ of degree n, there exists an isocomplex with $O(n^2)$ simplices.*

Proof. We consider the isocomplex returned by a cylindrical algebraic decomposition algorithm. It returns $O(n^2)$ many fibers of the curves (with respect to some projection direction) and connects the fiber points by straight-line segments. Since any fiber has at most n points, the complexity is $O(n^3)$. We can assume that no segment is vertical and consider the complex as a directed graph from left to right, with the fiber points as vertices. In particular, it makes sense to talk about the *in-degree* of a vertex as the number of edges that enter from the left hand side. We re-embed the graph into the plane with the following rules. (1) Each vertex remains the same x-coordinate, and the vertical ordering of the vertices at the same x-coordinate remains unchanged. (2) Each edge from a vertex of in-degree 1 to another vertex of in-degree 1 must be horizontal.

Properties (1) ensures that the result is isotopic to the original complex. A complex with properties (1) and (2) can be computed by a simple plane sweep algorithm (Fig. 3) Vertices adjacent to exactly two horizontal edges are removed afterwards, and the edges are merged. Let $\mathscr{C}_h$ denote this new complex. By construction, any maximal smooth x-monotone segment of the curve is represented by a polyline in $\mathscr{C}_h$ with two bends, running horizontally between the two bends. The number of edges is thus at most three times the number of segments of the curve that leave a critical point. Their number can be bounded by $O(n^2)$, thus the complexity of $\mathscr{C}_h$ is also $O(n^2)$. $\square$

4 Higher dimensions

We show to what extend our results for curves can be generalized into higher dimensions. Throughout this section, we consider $d \geq 2$ to be a fixed constant – this yields bounds of the form $\Omega/O(n^{h(d)})$ for some function h in d. However, one should keep in mind that the constants hidden in the O-notation depend on d. Furthermore, we still assume for simplicity that the considered hypersurface is bounded in each coordinate.

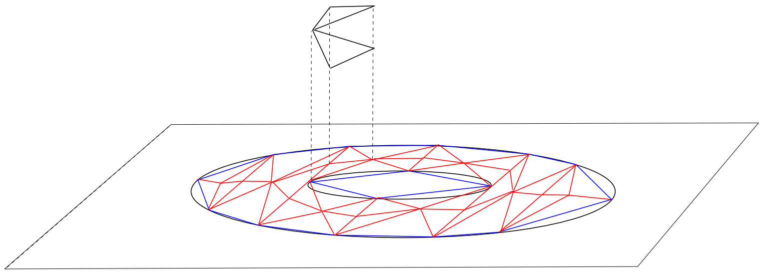

Fig. 4. Illustration of Proposition 4 for a torus: $\mathscr{O}_R$ is a plane curve consisting of 2 circles (in black). Its stable isocomplex is drawn in blue, and the completion in red color. Each triangle of the completion is lifted to $\mathbb{R}^3$; they form an isotopic triangulation of the torus.

Stable isocomplexes: The construction of Theorem 2 can be immediately transfered into arbitrary dimensions by choosing $O(n^d)$ isolated points close to $O(n)$ concentric d-spheres. This yields.

Proposition 3. *For any $n \in \mathbb{N}$ and $d \geq 2$, there exists an algebraic hypersurface $\mathscr{O} \subset \mathbb{R}^d$ of degree n such that any stable isocomplex for $\mathscr{O}$ has $\Omega(n^{d+1})$ vertices.*

This implies a lower bound of $\Omega(n^{d+1})$ for the total number of simplices, since a complex with v vertices can only have p to $2^d v$ many simplices.

Also the upper bound construction can be generalized; however, the exponent increases exponentially with d.

Proposition 4. *For a hypersurface $\mathscr{O} \subset \mathbb{R}^d$ of degree n, there exists a stable isocomplex with $O(n^{2^d-1})$ simplices.*

Proof. This result follows from the general theory on cylindrical algebraic decomposition. We only sketch the proof here. Let $f \in \mathbb{Q}[x_1, \ldots, x_d]$ be the equation for $\mathscr{O}$, then

$$R := \text{res}_{x_d}(f, \frac{\partial f}{\partial x_d}) \in \mathbb{Q}[x_1, \ldots, x_{d-1}]$$

defines a hypersurface $\mathscr{O}_R$ in $\mathbb{R}^{d-1}$. The algorithmic idea is to recursively build a simplicial complex for $\mathscr{O}_R$, and "completing" it to a simplicial complex that triangulates $\mathbb{R}^{d-1}$ (or more precisely, the projection of $\mathscr{O}$ on the first $d-1$ variables; see Fig. 4). The lifts of the cells of this completed simplicial complex form an isocomplex of $\mathscr{O}$ (this follows from the fact that $\mathscr{O}$ is *delineable* with respect to $\mathscr{O}_R$ [3, 5]).

The bound is proven by induction on d. The base case follows with Proposition 1. Assume that the claim is proven for $d-1$. Since $\mathscr{O}_R$ is of degree at most n^2, there is an isocomplex with $O(n^{2^d-2})$ simplices. Note that the simplices of this isocomplex are cylindrically arranged with respect to x_{d-1}. Thus, in order to complete the isocomplex, we only have to traingulate each cylinder, which can be done with a constant number of simplices per cylinder. Each cylinder can be assigned to the simplex of $\mathscr{O}_d$ below it. Therefore, the completed isocomplex has the same size as the isocomplex of $\mathscr{O}_R$. Since each of its simplices is lifted at most n times, the bound follows. □

General isocomplexes: Again, the simple lower bound from Proposition 2 transfers directly into higher dimensions by considering n hyperplanes in generic position:

Proposition 5. *For any $n \in \mathbb{N}$ and $d \geq 2$, there exists an algebraic hypersurface $\mathscr{O} \subset \mathbb{R}^d$ of degree n such that any isocomplex for $\mathscr{O}$ has $\Omega(n^d)$ vertices.*

The upper bound below follows using recursive projection as in Proposition 4 – the only difference is that the base case (the triangulation for $d = 2$) is replaced by the algorithm described in Theorem 3. Note that it is possible to complete this triangulation without increasing the complexity by constructing a trapezoidal decomposition and barycentrically subdividing the faces. We obtain:

Proposition 6. *For a hypersurface $\mathscr{O} \subset \mathbb{R}^d$ of degree n, there exists a stable isocomplex with $O(n^{3/4 \cdot 2^d - 1})$ cells.*

Not surprisingly, the improved base case lessens the growth of the isocomplex in higher dimensions, but due to the projection strategy used in the construction, the bound remains double exponential in d. We believe that these upper bounds are not tight – it might be possible to improve them by a triangulation method not based on projection. However, already for algebraic surfaces, it seems difficult to come up with a simplification algorithm which provably reduces the complexity and preserves the topology at the same time.

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