# SLD-Resolution Reduction of Second-Order Horn Fragments

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## SLD-Resolution Reduction of Second-Order Horn Fragments

Prolog-based systems

- on Horn clauses
- using SLD-resolution (Selective Linear Definite)
  - sound and refutationally complete on Horn clauses
  - without factorization of literals
  - here duplication of literals is forbidden (implies loss of refutational completeness)

# SLD-Resolution Reduction of Second-Order Horn Fragments

## Prolog-based Inductive Logic Programming (ILP) systems

**Input**

({Parent_of(Ana,Ben),Parent_of(Ben,Carol)},Grand_parent_of(Ana,Carol))
({Parent_of(Arthur,Betty),Parent_of(Betty,Charlie)},Grand_parent_of(Arthur,Charlie))

**ILP**

**Metarules**

$\vdots$

$P_0(A,C) \leftarrow P_1(A,B), P_2(B,C)$

$P_3(A) \leftarrow P_4(A), P_5(A)$

$P_0(A,B) \leftarrow P_1(A,B), P_3(B)$

$\vdots$

**Output**

Grand_parent_of$(A,C) \leftarrow$ Parent_of$(A,B)$,Parent_of$(B,C)$

# SLD-Resolution Reduction of Second-Order Horn Fragments

**Second-Order Horn Fragment $\mathcal{H}$**

$P_0(A) \leftarrow P_1(A, B), P_2(C, C)$ [not interesting]

**Connected Fragment $\mathcal{H}^c$**

$P_0(A) \leftarrow P_1(A, B), P_2(B, C)$ [mildly interesting]

**2-Connected Fragment $\mathcal{H}^{2c}$**

$P_0(A, C) \leftarrow P_1(A, B), P_2(B, C), P_4(A)$ [very interesting]

## SLD-Resolution **Reduction** of Second-Order Horn Fragments

What is the best set of metarules to use?

- Describes the desired fragment completely.
- Does not take too much memory.
- Allows for an efficient exploration of the search space

Can we reduce a fragment to a finite subset with these properties?

# First Idea: Entailment Reduction

[Cropper, Muggleton, ILP'14]

$$\begin{aligned}
C_1 &= P_0(A,B) \leftarrow P_1(A,B)\\
C_2 &= P_0(A,B) \leftarrow P_1(A,B), P_2(A)\\
C_3 &= P_0(A,B) \leftarrow P_1(A,B), P_3(A,B)\\
C_4 &= P_0(A,B) \leftarrow P_1(A,B), P_3(A,B), P_4(A,B)
\end{aligned}$$

$$\{C_1\} \models \{C_1, C_2, C_3, C_4\}$$

## Loss of completeness

$$C_1 \not\vdash_{SLD} C_2, C_3, C_4$$

## Better Idea: Derivation Reduction

$$
\begin{aligned}
C_1 &= P_0(A, B) \leftarrow P_1(A, B)\\
C_2 &= P_0(A, B) \leftarrow P_1(A, B), P_2(A)\\
C_3 &= P_0(A, B) \leftarrow P_1(A, B), P_3(A, B)\\
C_4 &= P_0(A, B) \leftarrow P_1(A, B), P_3(A, B), P_4(A, B)
\end{aligned}
$$

$$\{C_1, C_2, C_3\} \vdash_{\mathrm{SLD}} \{C_1, C_2, C_3, C_4\}$$

This problem is undecidable!

Can this be done for the fragments of interest?

## Reduction of Connected Fragments

Given a fragment $\mathcal{F}$, the fragment $\mathcal{F}_{a,b}$ is such that:

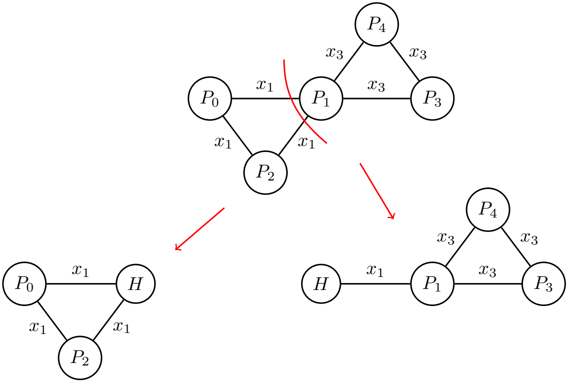
- $a$ is the maximal arity of the predicates,
- $b$ is the maximal number of literals in the body of clauses,
- $\infty$ means unbounded.
- $\mathcal{F}$ is reducible from $\mathcal{F}'$ if any clause in $\mathcal{F}$ can be derived using SLD-resolution from clauses in $\mathcal{F}'$.

$\forall a \in \mathbb{N}^*, \mathcal{H}^c_{a,\infty}$ is reducible to $\mathcal{H}^c_{a,2}$.

The fragment $\mathcal{H}^c$ is reducible to $\mathcal{H}^c_{\infty,2}$.

# Reduction of the Connected Fragment $\mathcal{H}^{c}_{2,\infty}$

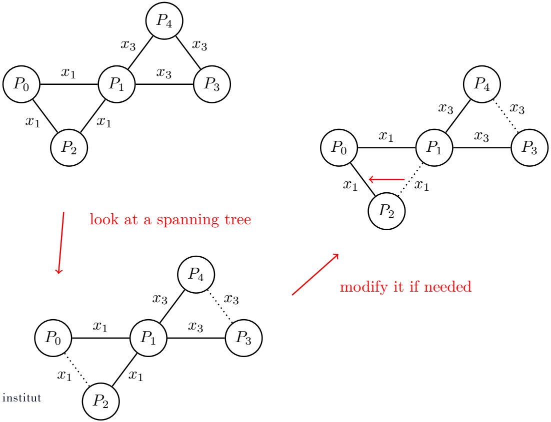
$$P_0(x_1, x_2) \leftarrow P_1(x_3, x_1), P_2(x_1), P_3(x_3), P_4(x_3)$$




$$P_0(x_1, x_2) \leftarrow P_2(x_1), H(x_1) \qquad H(x_1) \leftarrow P_1(x_3, x_1), P_3(x_3), P_4(x_3)$$

# Proof Idea of the Reduction of Connected Fragments

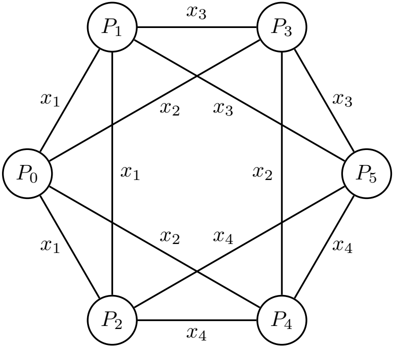
find a spanning tree where two adjacent vertices have at most *a* outgoing edges [here $a = 2$]



# ~~Reduction~~ of the 2-Connected Fragment $\mathcal{H}^{2c}_{2,\infty}$

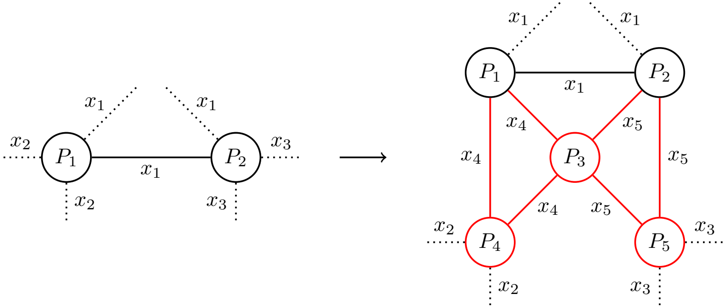
Not possible

# Counter-example for $\mathcal{H}^{2c}_{2,5}$




$$P_0(x_1, x_2) \leftarrow P_1(x_1, x_3), P_2(x_1, x_4), P_3(x_2, x_3), P_4(x_2, x_4), P_5(x_3, x_4)$$

# Counter-example for $\mathcal{H}^{2c}_{2,\infty}$




This transformation preserves irreducibility while increasing the size of the clause.

# Summary

| | SLD-resolution | resolution |
|---|---|---|
| connected ($\mathcal{H}^c$) | $\mathcal{H}^c_{\infty,2}$ | $\mathcal{H}^c_{\infty,2}$ |
| 2-connected ($\mathcal{H}^{2c}_{2,\infty}$) | NO | $\mathcal{H}^{2c}_{2,2}$ |

# Counter-measures for the 2-Connected Fragment $\mathcal{H}^{2c}_{2,\infty}$

- Use standard resolution
- Allow a restricted use of triadic predicates
- Add irreducible clauses dynamically
- ... ?