## 2.4 Algorithmic Problems

**Validity($F$):** $\models F$ ?

**Satisfiability($F$):** $F$ satisfiable?

**Entailment($F$,$G$):** does $F$ entail $G$?

**Model($A$,$F$):** $A \models F$?

**Solve($A$,$F$):** find an assignment $\beta$ such that $A, \beta \models F$

**Solve($F$):** find a substitution $\sigma$ such that $\models F\sigma$

**Abduce($F$):** find $G$ with "certain properties" such that $G$ entails $F$



## Gödel's Famous Theorems

1. For most signatures $\Sigma$, validity is undecidable for $\Sigma$-formulas. (One can easily encode Turing machines in most signatures.)
2. For each signature $\Sigma$, the set of valid $\Sigma$-formulas is recursively enumerable. (We will prove this by giving complete deduction systems.)
3. For $\Sigma = \Sigma_{PA}$ and $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the theory $Th(\mathbb{N}_*)$ is not recursively enumerable.

These complexity results motivate the study of subclasses of formulas (fragments) of first-order logic

$Q$: Can you think of any fragments of first-order logic for which validity is decidable?



## Some Decidable Fragments

Some decidable fragments:

- Monadic class: no function symbols, all predicates unary; validity is NEXPTIME-complete.
- Variable-free formulas without equality: satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- Finite model checking is decidable in time polynomial in the size of the structure and the formula.



## 2.5 Normal Forms and Skolemization

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.



## Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \; F,$$

where $F$ is quantifier-free and $Q_i \in \{\forall, \exists\}$;
we call $Q_1 x_1 \ldots Q_n x_n$ the quantifier prefix and $F$ the matrix of the formula.



## Prenex Normal Form

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$\begin{array}{rcll}
(F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \wedge (G \rightarrow F) & \\
\neg Q x F & \Rightarrow_P & \overline{Q} x \neg F & (\neg Q) \\
(QxF \;\rho\; G) & \Rightarrow_P & Qy(F[y/x] \;\rho\; G),\ y \text{ fresh},\ \rho \in \{\wedge, \vee\} & \\
(QxF \rightarrow G) & \Rightarrow_P & \overline{Q}y(F[y/x] \rightarrow G),\ y \text{ fresh} & \\
(F \;\rho\; QxG) & \Rightarrow_P & Qy(F \;\rho\; G[y/x]),\ y \text{ fresh},\ \rho \in \{\wedge, \vee, \rightarrow\} &
\end{array}$$

Here $\overline{Q}$ denotes the quantifier dual to $Q$, i.e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.



## Skolemization

**Intuition:** replacement of $\exists y$ by a concrete choice function computing $y$ from all the arguments $y$ depends on.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \ldots, x_n \exists y F \quad \Rightarrow_S \quad \forall x_1, \ldots, x_n F[f(x_1, \ldots, x_n)/y]$$

where $f/n$ is a new function symbol (Skolem function).



## Skolemization

**Together:** $F \Rightarrow_P^* \underbrace{G}_{\text{prenex}} \Rightarrow_S^* \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 2.9
Let $F$, $G$, and $H$ as defined above and closed. Then

(i) $F$ and $G$ are equivalent.

(ii) $H \models G$ but the converse is not true in general.

(iii) $G$ satisfiable (wrt. $\Sigma$-Alg) $\Leftrightarrow$ $H$ satisfiable (wrt. $\Sigma'$-Alg) where $\Sigma' = (\Omega \cup SKF, \Pi)$, if $\Sigma = (\Omega, \Pi)$.

## Clausal Normal Form (Conjunctive Normal Form)

$$
\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \wedge (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \vee G) \\
\neg(F \vee G) & \Rightarrow_K & (\neg F \wedge \neg G) \\
\neg(F \wedge G) & \Rightarrow_K & (\neg F \vee \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \wedge G) \vee H & \Rightarrow_K & (F \vee H) \wedge (G \vee H) \\
(F \wedge \top) & \Rightarrow_K & F \\
(F \wedge \bot) & \Rightarrow_K & \bot \\
(F \vee \top) & \Rightarrow_K & \top \\
(F \vee \bot) & \Rightarrow_K & F
\end{array}
$$

These rules are to be applied modulo associativity and commutativity of $\wedge$ and $\vee$. The first five rules, plus the rule $(\neg Q)$, compute the negation normal form (NNF) of a formula.



## The Complete Picture

$$
\begin{array}{llll}
F & \Rightarrow^*_P & Q_1 y_1 \dots Q_n y_n \; G & (G \text{ quantifier-free}) \\
 & \Rightarrow^*_S & \forall x_1, \dots, x_m \; H & (m \leq n,\ H \text{ quantifier-free}) \\
 & \Rightarrow^*_K & \underbrace{\underbrace{\forall x_1, \dots, x_m}_{\text{leave out}} \; \bigwedge_{i=1}^{k} \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'} &
\end{array}
$$

$N = \{C_1, \dots, C_k\}$ is called the clausal (normal) form (CNF) of $F$.
*Note:* the variables in the clauses are implicitly universally quantified.



## The Complete Picture

Theorem 2.10
Let $F$ be closed. Then $F' \models F$.
(The converse is not true in general.)

Theorem 2.11
Let $F$ be closed. Then $F$ is satisfiable iff $F'$ is satisfiable iff $N$ is satisfiable



## Optimization

Here is lots of room for optimization since we only can preserve satisfiability anyway:

- size of the CNF exponential when done naively;
- want to preserve the original formula structure;
- want small arity of Skolem functions.



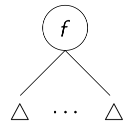
## 2.6 Herbrand Interpretations

From now an we shall consider PL without equality. $\Omega$ shall contains at least one constant symbol.

A Herbrand interpretation (over $\Sigma$) is a $\Sigma$-algebra $\mathcal{A}$ such that

- $U_\mathcal{A} = \mathrm{T}_\Sigma$ (= the set of ground terms over $\Sigma$)
- $f_\mathcal{A} : (s_1, \dots, s_n) \mapsto f(s_1, \dots, s_n)$, $f/n \in \Omega$

$f_\mathcal{A}(\triangle, \dots, \triangle) =$






## Herbrand Interpretations

In other words, *values are fixed* to be ground terms and *functions are fixed* to be the term constructors. Only predicate symbols $p/m \in \Pi$ may be freely interpreted as relations $p_\mathcal{A} \subseteq \mathrm{T}^m_\Sigma$.

Proposition 2.12
Every set of ground atoms $I$ uniquely determines a Herbrand interpretation $\mathcal{A}$ via

$$(s_1, \dots, s_n) \in p_\mathcal{A} \quad :\Leftrightarrow \quad p(s_1, \dots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over $\Sigma$) with sets of $\Sigma$-ground atoms.



## Herbrand Interpretations

*Example:* $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over $\Sigma_{Pres}$:

$$
\begin{array}{ll}
I = \{ & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \dots, \\
 & 0 + 0 \leq 0,\ 0 + 0 \leq s(0),\ \dots, \\
 & \dots,\ (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\
 & \dots \\
 & s(0) + 0 < s(0) + 0 + 0 + s(0) \\
 & \dots\}
\end{array}
$$



## Existence of Herbrand Models

A Herbrand interpretation $I$ is called a Herbrand model of $F$, if $I \models F$.

Theorem 2.13
Let $N$ be a set of $\Sigma$-clauses.

$$
\begin{array}{lcl}
N \text{ satisfiable} & \Leftrightarrow & N \text{ has a Herbrand model (over } \Sigma) \\
 & \Leftrightarrow & G_\Sigma(N) \text{ has a Herbrand model (over } \Sigma)
\end{array}
$$

where $G_\Sigma(N) = \{C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \rightarrow \mathrm{T}_\Sigma\}$ is the set of ground instances of $N$.

[The proof will be given below in the context of the completeness proof for resolution.]

## Example of a $G_\Sigma$

For $\Sigma_{Pres}$ one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$
$(s(0) < 0) \vee (0 \leq s(s(0)))$
...
$(s(0) + s(0) < s(0) + 0) \vee (s(0) + 0 \leq s(s(0) + s(0)))$
...



## 2.7 Inference Systems and Proofs

Inference systems $\Gamma$ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}),\ \ n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \ \ldots \ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\ .$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures (cf. below).



## Proofs

A proof in $\Gamma$ of a formula $F$ from a a set of formulas $N$ (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ldots, F_{i_{n_i}}, F_i)$ in $\Gamma$, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.



## Soundness and Completeness

Provability $\vdash_\Gamma$ of $F$ from $N$ in $\Gamma$:
$N \vdash_\Gamma F \ \ :\Leftrightarrow$ there exists a proof $\Gamma$ of $F$ from $N$.

$\Gamma$ is called sound $:\Leftrightarrow$

$$\frac{F_1 \ \ldots \ F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \ldots, F_n \models F$$

$\Gamma$ is called complete $:\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

$\Gamma$ is called refutationally complete $:\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$



## Soundness and Completeness

Proposition 2.14

(i) Let $\Gamma$ be sound. Then $N \vdash_\Gamma F \Rightarrow N \models F$

(ii) $N \vdash_\Gamma F \Rightarrow$ there exist $F_1, \ldots, F_n \in N$ s.t. $F_1, \ldots, F_n \vdash_\Gamma F$ (resembles compactness).



## Proofs as Trees

| | | | | |
|---|---|---|---|---|
| markings | $\widehat{=}$ | formulas | | |
| leaves | $\widehat{=}$ | assumptions and axioms | | |
| other nodes | $\widehat{=}$ | inferences: conclusion | $\widehat{=}$ | ancestor |
| | | premises | $\widehat{=}$ | direct descendants |

$$\dfrac{P(g(a,b)) \qquad \dfrac{\dfrac{\dfrac{P(f(a)) \vee Q(b) \qquad \dfrac{\dfrac{P(f(a)) \vee Q(b) \quad \neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)}{\neg P(f(a)) \vee Q(b) \vee Q(b)}}{\neg P(f(a)) \vee Q(b)}}{Q(b) \vee Q(b)}}{Q(b)} \qquad \neg P(f(a)) \vee \neg Q(b)}{\neg P(g(a,b))}}{\bot}$$



## 2.8 Propositional Resolution

We observe that propositional clauses and ground clauses are the same concept.

In this section we only deal with ground clauses.



## The Resolution Calculus *Res*

Resolution inference rule:

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

Terminology: $C \vee D$: resolvent; $A$: resolved atom

(Positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

## The Resolution Calculus *Res*

These are schematic inference rules; for each substitution of the schematic variables $C$, $D$, and $A$, respectively, by ground clauses and ground atoms we obtain an inference rule.

As "$\vee$" is considered associative and commutative, we assume that $A$ and $\neg A$ can occur anywhere in their respective clauses.



## Sample Refutation

| | | |
|---|---|---|
| 1. | $\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$ | (given) |
| 2. | $P(f(a)) \vee Q(b)$ | (given) |
| 3. | $\neg P(g(b,a)) \vee \neg Q(b)$ | (given) |
| 4. | $P(g(b,a))$ | (given) |
| 5. | $\neg P(f(a)) \vee Q(b) \vee Q(b)$ | (Res. 2. into 1.) |
| 6. | $\neg P(f(a)) \vee Q(b)$ | (Fact. 5.) |
| 7. | $Q(b) \vee Q(b)$ | (Res. 2. into 6.) |
| 8. | $Q(b)$ | (Fact. 7.) |
| 9. | $\neg P(g(b,a))$ | (Res. 8. into 3.) |
| 10. | $\perp$ | (Res. 4. into 9.) |



## Resolution with Implicit Factorization *RIF*

$$\frac{C \vee A \vee \ldots \vee A \qquad \neg A \vee D}{C \vee D}$$

| | | |
|---|---|---|
| 1. | $\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$ | (given) |
| 2. | $P(f(a)) \vee Q(b)$ | (given) |
| 3. | $\neg P(g(b,a)) \vee \neg Q(b)$ | (given) |
| 4. | $P(g(b,a))$ | (given) |
| 5. | $\neg P(f(a)) \vee Q(b) \vee Q(b)$ | (Res. 2. into 1.) |
| 6. | $Q(b) \vee Q(b) \vee Q(b)$ | (Res. 2. into 5.) |
| 7. | $\neg P(g(b,a))$ | (Res. 6. into 3.) |
| 8. | $\perp$ | (Res. 4. into 7.) |



## Soundness of Resolution

Theorem 2.15
Propositional resolution is sound.

Proof:
Let $I \in \Sigma$-Alg. To be shown:

(i) for resolution: $I \models C \vee A,\ I \models D \vee \neg A \ \Rightarrow\ I \models C \vee D$

(ii) for factorization: $I \models C \vee A \vee A \ \Rightarrow\ I \models C \vee A$

ad (i): Assume premises are valid in $I$. Two cases need to be considered: (a) $A$ is valid, or (b) $\neg A$ is valid.
a) $I \models A \Rightarrow I \models D \Rightarrow I \models C \vee D$
b) $I \models \neg A \Rightarrow I \models C \Rightarrow I \models C \vee D$
ad (ii): even simpler.



## Soundness of Resolution

Note: In propositional logic (ground clauses) we have:

1. $I \models L_1 \vee \ldots \vee L_n \ \Leftrightarrow\ $ there exists $i$: $I \models L_i$.
2. $I \models A$ or $I \models \neg A$.