2.10 Refutational Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$, or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e.g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

$$\begin{array}{lcll} [\neg]A & \succ_L & [\neg]B & \text{, if } A \succ B \\ \neg A & \succ_L & A & \end{array}$$

3. Extend $\succ_L$ to an ordering $\succ_C$ on ground clauses: $\succ_C = (\succ_L)_{\mathrm{mul}}$, the multi-set extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$. Then:

$$\begin{array}{cc} & A_0 \lor A_1 \\ \prec & A_1 \lor A_2 \\ \prec & \neg A_1 \lor A_2 \\ \prec & \neg A_1 \lor A_4 \lor A_3 \\ \prec & \neg A_1 \lor \neg A_4 \lor A_3 \\ \prec & \neg A_5 \lor A_5 \end{array}$$

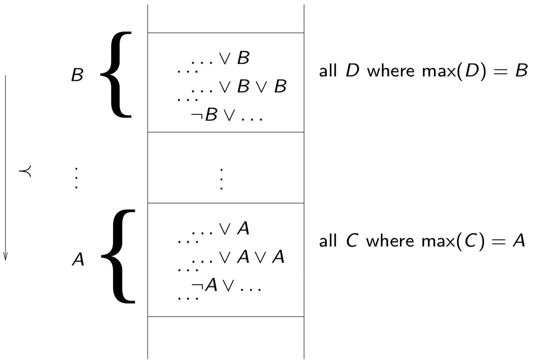
Properties of the Clause Ordering

Proposition 2.20:

1. The orderings on literals and clauses are total and well-founded.
2. Let C and D be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in C.
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$, A occurs negatively in C but only positively in D, then $C \succ D$.

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:



Closure of Clause Sets under *Res*

$$\begin{aligned} Res(N) &= \{C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N\} \\ Res^0(N) &= N \\ Res^{n+1}(N) &= Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0 \\ Res^*(N) &= \textstyle\bigcup_{n\geq 0} Res^n(N) \end{aligned}$$

N is called saturated (wrt. resolution), if $Res(N) \subseteq N$.

Proposition 2.21:

(i) $Res^*(N)$ is saturated.

(ii) Res is refutationally complete, iff for each set N of ground clauses:

$$N \models \bot \;\Leftrightarrow\; \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of ground clauses, atom ordering $\succ$.
Wanted: Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- $I \models N$, *if N is saturated and* $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.

Example

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C
2	$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
3	$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	true in I_C
4	$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
5	$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
6	$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not maximal; *min. counter-ex.*
7	$\neg A_1 \lor A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a counterexample.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.

Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).

Resolution Reduces Counterexamples

$$\frac{\neg A_1 \lor A_4 \lor A_3 \lor A_0 \quad \neg A_1 \lor \neg A_4 \lor A_3}{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *minimal counter-ex.*
$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_4\}$	
$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	counterexample
$\neg A_1 \lor A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

$$\frac{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0}{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_3 \lor A_5$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Construction of Candidate Models Formally

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \lor A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

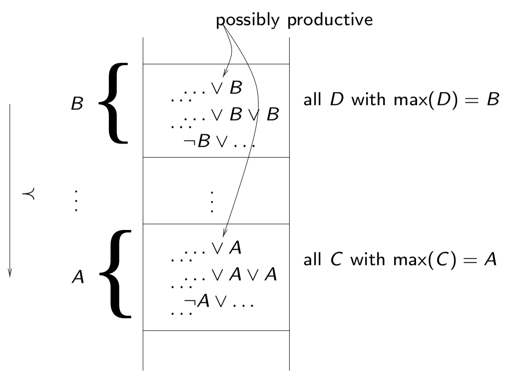
We say that C produces A, if $\Delta_C = \{A\}$.

The candidate model for N (wrt. $\succ$) is given as $I_N^\succ := \bigcup_C \Delta_C$.

We also simply write I_N, or I, for $I_N^\succ$ if $\succ$ is either irrelevant or known from the context.

Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



Some Properties of the Construction

Proposition 2.22:

(i) $C = \neg A \lor C' \ \Rightarrow\ $ no $D \succeq C$ produces A.

(ii) C productive $\Rightarrow I_C \cup \Delta_C \models C$.

(iii) Let $D' \succ D \succeq C$. Then

$$I_D \cup \Delta_D \models C \Rightarrow I_{D'} \cup \Delta_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$:

$$I_D \cup \Delta_D \not\models C \Rightarrow I_{D'} \cup \Delta_{D'} \not\models C \text{ and } I_N \not\models C.$$

Some Properties of the Construction

(iv) Let $D' \succ D \succ C$. Then

$$I_D \models C \Rightarrow I_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$:

$$I_D \not\models C \Rightarrow I_{D'} \not\models C \text{ and } I_N \not\models C.$$

(v) $D = C \lor A$ produces $A \Rightarrow I_N \not\models C$.

Model Existence Theorem

Theorem 2.23 (Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

Corollary 2.24:
Let N be saturated wrt. Res. Then $N \models \bot \Leftrightarrow \bot \in N$.

Model Existence Theorem

Proof of Theorem 2.23:
Suppose $\bot \notin N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^{\succ} \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$ (i.e., the maximal atom occurs negatively)
$\Rightarrow I_N \models A$ and $I_N \not\models C'$
$\Rightarrow$ some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$
$\Rightarrow$ contradicts minimality of C.

Case 2: $C = C' \vee A \vee A$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller counterexample $C' \vee A \in N$. $\Rightarrow$ contradicts minimality of C.

Compactness of Propositional Logic

Theorem 2.25 (Compactness):
Let N be a set of propositional formulas. Then N is unsatisfiable, if and only if some finite subset $M \subseteq N$ is unsatisfiable.

Proof:
"$\Leftarrow$": trivial.

"$\Rightarrow$": Let N be unsatisfiable.
$\Rightarrow Res^*(N)$ unsatisfiable
$\Rightarrow \bot \in Res^*(N)$ by refutational completeness of resolution
$\Rightarrow \exists n \geq 0 : \bot \in Res^n(N)$
$\Rightarrow \bot$ has a finite resolution proof P;
choose M as the set of assumptions in P.