2.11 General Resolution

Propositional resolution:

refutationally complete,

in its most naive version:
not guaranteed to terminate for satisfiable sets of clauses,
(improved versions do terminate, however)

clearly inferior to the DPLL procedure
(even with various improvements).

But: in contrast to the DPLL procedure, resolution can be easily extended to non-ground clauses.

General Resolution through Instantiation

Idea: instantiate clauses appropriately:

$$
\begin{tikzcd}
P(z',z') \vee \neg Q(z) \arrow[d, "{[a/z', f(a,b)/z]}"'] & & & \neg P(a,y) \arrow[dl, "{[a/y]}"'] \arrow[dr, "{[b/y]}"] & & & P(x',b) \vee Q(f(x',x)) \arrow[d, "{[a/x', b/x]}"] \\
P(a,a) \vee \neg Q(f(a,b)) \arrow[dr] & & \neg P(a,a) \arrow[dl] & & \neg P(a,b) \arrow[dr] & & P(a,b) \vee Q(f(a,b)) \arrow[dl] \\
& \neg Q(f(a,b)) \arrow[d] & & & & Q(f(a,b)) \arrow[d] & \\
& {} \arrow[drr] & & & & {} \arrow[dll] & \\
& & & \bot & & &
\end{tikzcd}
$$

General Resolution through Instantiation

Problems:

More than one instance of a clause can participate in a proof.

Even worse: There are infinitely many possible instances.

Observation:

Instantiation must produce complementary literals
(so that inferences become possible).

Idea:

Do not instantiate more than necessary to get complementary literals.

General Resolution through Instantiation

Idea: do not instantiate more than necessary:

$$
\begin{tikzcd}
P(z',z') \vee \neg Q(z) \arrow[d, "{[a/z']}"'] & & & \neg P(a,y) \arrow[dl, "{[a/y]}"'] \arrow[dr, "{[b/y]}"] & & & P(x',b) \vee Q(f(x',x)) \arrow[d, "{[a/x']}"] \\
P(a,a) \vee \neg Q(z) \arrow[dr] & & \neg P(a,a) \arrow[dl] & & \neg P(a,b) \arrow[dr] & & P(a,b) \vee Q(f(a,x)) \arrow[dl] \\
& \neg Q(z) \arrow[d, "{[f(a,x)/z]}"'] & & & & Q(f(a,x)) \arrow[d] & \\
& \neg Q(f(a,x)) \arrow[drr] & & & & Q(f(a,x)) \arrow[dll] & \\
& & & \bot & & &
\end{tikzcd}
$$

Lifting Principle

Problem: Make saturation of infinite sets of clauses as they arise from taking the (ground) instances of finitely many general clauses (with variables) effective and efficient.

Idea (Robinson 65):

- Resolution for general clauses:
- *Equality* of ground atoms is generalized to *unifiability* of general atoms;
- Only compute *most general* (minimal) unifiers.

Lifting Principle

Significance: The advantage of the method in (Robinson 65) compared with (Gilmore 60) is that unification enumerates only those instances of clauses that participate in an inference. Moreover, clauses are not right away instantiated into ground clauses. Rather they are instantiated only as far as required for an inference. Inferences with non-ground clauses in general represent infinite sets of ground inferences which are computed simultaneously in a single step.

Resolution for General Clauses

General binary resolution *Res*:

$$\frac{C \vee A \qquad D \vee \neg B}{(C \vee D)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A, B) \qquad [\text{resolution}]$$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A, B) \quad [\text{factorization}]$$

General resolution *RIF* **with implicit factorization:**

$$\frac{C \vee A_1 \vee \ldots \vee A_n \qquad D \vee \neg B}{(C \vee D)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A_1, \ldots, A_n, B)$$

[RIF]

Resolution for General Clauses

For inferences with more than one premise, we assume that the variables in the premises are (bijectively) renamed such that they become different to any variable in the other premises. We do not formalize this. Which names one uses for variables is otherwise irrelevant.

Unification

Let $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ (s_i, t_i terms or atoms) a multi-set of equality problems. A substitution σ is called a unifier of E if $s_i\sigma = t_i\sigma$ for all $1 \leq i \leq n$.

If a unifier of E exists, then E is called unifiable.

Unification

A substitution σ is called more general than a substitution τ, denoted by $\sigma \leq \tau$, if there exists a substitution ρ such that $\rho \circ \sigma = \tau$, where $(\rho \circ \sigma)(x) := (x\sigma)\rho$ is the composition of σ and ρ as mappings.
(Note that $\rho \circ \sigma$ has a finite domain as required for a substitution.)

If a unifier of E is more general than any other unifier of E, then we speak of a most general unifier of E, denoted by $\mathrm{mgu}(E)$.

Unification

Proposition 2.26:

(i) $\leq$ is a quasi-ordering on substitutions, and $\circ$ is associative.

(ii) If $\sigma \leq \tau$ and $\tau \leq \sigma$ (we write $\sigma \sim \tau$ in this case), then $x\sigma$ and $x\tau$ are equal up to (bijective) variable renaming, for any x in X.

A substitution σ is called idempotent, if $\sigma \circ \sigma = \sigma$.

Proposition 2.27:
σ is idempotent iff $\mathit{dom}(\sigma) \cap \mathit{codom}(\sigma) = \emptyset$.

Unification after Martelli/Montanari

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{MM} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{MM} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{MM} & \bot \\
x \doteq t, E & \Rightarrow_{MM} & x \doteq t, E[t/x] \\
 & & \text{if } x \in \mathit{var}(E), x \notin \mathit{var}(t) \\
x \doteq t, E & \Rightarrow_{MM} & \bot \\
 & & \text{if } x \neq t, x \in \mathit{var}(t) \\
t \doteq x, E & \Rightarrow_{MM} & x \doteq t, E \\
 & & \text{if } t \notin X
\end{array}
$$

MM: Main Properties

If $E = x_1 \doteq u_1, \ldots, x_k \doteq u_k$, with x_i pairwise distinct, $x_i \notin var(u_j)$, then E is called an (equational problem in) solved form representing the solution $\sigma_E = [u_1/x_1, \ldots, u_k/x_k]$.

Proposition 2.28:
If E is a solved form then σ_E is am mgu of E.

MM: Main Properties

Theorem 2.29:

1. If $E \Rightarrow_{MM} E'$ then σ is a unifier of E iff σ is a unifier of E'
2. If $E \Rightarrow^*_{MM} \bot$ then E is not unifiable.
3. If $E \Rightarrow^*_{MM} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.

Proof:
(1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ [t/x] = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u[t/x]\sigma = v[t/x]\sigma$. (2) and (3) follow by induction from (1) using Proposition 2.28.

Main Unification Theorem

Theorem 2.30:
E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$.

Problem: *exponential growth* of terms possible

Main Unification Theorem

Proof of Theorem 2.30:

- $\Rightarrow_{MM}$ is Noetherian. A suitable lexicographic ordering on the multisets E (with $\bot$ minimal) shows this. Compare in this order:
  1. the number of defined variables (d.h. variables x in equations $x \doteq t$ with $x \notin var(t)$), which also occur outside their definition elsewhere in E;
  2. the multi-set ordering induced by (i) the size (number of symbols) in an equation; (ii) if sizes are equal consider $x \doteq t$ smaller than $t \doteq x$, if $t \notin X$.

Main Unification Theorem

- A system E that is irreducible wrt. $\Rightarrow_{MM}$ is either $\bot$ or a solved form.
- Therefore, reducing any E by MM will end (no matter what reduction strategy we apply) in an irreducible E' having the same unifiers as E, and we can read off the mgu (or non-unifiability) of E from E' (Theorem 2.29, Proposition 2.28).
- σ is idempotent because of the substitution in rule 4. $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$, as no new variables are generated.

Lifting Lemma

Lemma 2.31:
Let C and D be variable-disjoint clauses. If

$$\begin{array}{ccc} C & & D \\ \downarrow \sigma & & \downarrow \rho \\ \end{array}$$

$$\frac{C\sigma \qquad D\rho}{C'} \qquad \text{[propositional resolution]}$$

then there exists a substitution τ such that

$$\frac{C \qquad D}{C''} \qquad \text{[general resolution]}$$

$$\downarrow \tau$$

$$C' = C''\tau$$

Lifting Lemma

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 2.32:
Let N be a set of general clauses saturated under Res, i.e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Saturation of Sets of General Clauses

Proof:
W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $C\sigma$ and $D\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, C and D are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar.

Herbrand's Theorem

Lemma 2.33:
Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation.
Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.

Lemma 2.34:
Let N be a set of Σ-clauses, let $\mathcal{A}$ be a *Herbrand* interpretation.
Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.

Herbrand's Theorem

Theorem 2.35 (Herbrand):
A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$
\begin{aligned}
N \not\models \bot &\Rightarrow \bot \notin Res^*(N) && \text{(resolution is sound)}\\
&\Rightarrow \bot \notin G_\Sigma(Res^*(N))\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models G_\Sigma(Res^*(N)) && \text{(Thm. 2.23; Cor. 2.32)}\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models Res^*(N) && \text{(Lemma 2.34)}\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models N && (N \subseteq Res^*(N))
\end{aligned}
$$

The Theorem of Löwenheim-Skolem

Theorem 2.36 (Löwenheim–Skolem):
Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.

Proof:
If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 2.35.

Refutational Completeness of General Resolution

Theorem 2.37:
Let N be a set of general clauses where $Res(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N.$$

Proof:
Let $Res(N) \subseteq N$. By Corollary 2.32: $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$$\begin{aligned} N \models \bot &\Leftrightarrow G_\Sigma(N) \models \bot && \text{(Lemma 2.33/2.34; Theorem 2.35)} \\ &\Leftrightarrow \bot \in G_\Sigma(N) && \text{(propositional resolution sound and complete)} \\ &\Leftrightarrow \bot \in N \end{aligned}$$

Compactness of Predicate Logic

Theorem 2.38 (Compactness Theorem for First-Order Logic):
Let Φ be a set of first-order formulas.
Φ is unsatisfiable $\Leftrightarrow$ some finite subset $\Psi \subseteq \Phi$ is unsatisfiable.

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let Φ be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in Φ. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 2.37, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose Ψ as the subset of formulas in Φ such that the corresponding clauses contain the assumptions (leaves) of B.