## 2.12 Ordered Resolution with Selection

Motivation: Search space for *Res* *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 2.23) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *order restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

# Selection Functions

A selection function is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \lor \neg A \lor B$$

$$\boxed{\neg B_0} \lor \boxed{\neg B_1} \lor A$$

## Resolution Calculus $Res_S^{\succ}$

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

Let $\succ$ be a total and well-founded ordering on ground atoms. A literal $L$ is called [strictly] maximal in a clause $C$ if and only if there exists a ground substitution $\sigma$ such that for all $L'$ in $C$: $L\sigma \succeq L'\sigma$ $[L\sigma \succ L'\sigma]$.

# Resolution Calculus $Res_S^{\succ}$

Let $\succ$ be an atom ordering and $S$ a selection function.

$$\frac{C \vee A \qquad \neg B \vee D}{(C \vee D)\sigma} \qquad \text{[ordered resolution with selection]}$$

if $\sigma = \mathrm{mgu}(A, B)$ and

(i) $A\sigma$ strictly maximal wrt. $C\sigma$;

(ii) nothing is selected in $C$ by $S$;

(iii) either $\neg B$ is selected,
or else nothing is selected in $\neg B \vee D$ and $\neg B\sigma$ is maximal in $D\sigma$.

## Resolution Calculus $Res_S^{\succ}$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad \text{[ordered factoring]}$$

if $\sigma = \mathrm{mgu}(A, B)$ and $A\sigma$ is maximal in $C\sigma$ and nothing is selected in $C$.

## Special Case: Propositional Logic

For ground clauses the resolution inference simplifies to

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D}$$

if

(i) $A \succ C$;

(ii) nothing is selected in $C$ by. S;

(iii) $\neg A$ is selected in $D \vee \neg A$,
or else nothing is selected in $D \vee \neg A$ and $\neg A \succeq \max(D)$.

Note: For positive literals, $A \succ C$ is the same as $A \succ \max(C)$.

# Search Spaces Become Smaller

| | | |
|---|---|---|
| 1 | $A \lor B$ | |
| 2 | $A \lor \boxed{\neg B}$ | |
| 3 | $\neg A \lor B$ | |
| 4 | $\neg A \lor \boxed{\neg B}$ | |
| 5 | $B \lor B$ | Res 1, 3 |
| 6 | $B$ | Fact 5 |
| 7 | $\neg A$ | Res 6, 4 |
| 8 | $A$ | Res 6, 2 |
| 9 | $\perp$ | Res 8, 7 |

we assume $A \succ B$ and $S$ as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

# Avoiding Rotation Redundancy

From

$$\dfrac{\dfrac{C_1 \vee A \quad C_2 \vee \neg A \vee B}{C_1 \vee C_2 \vee B} \qquad C_3 \vee \neg B}{C_1 \vee C_2 \vee C_3}$$

we can obtain by rotation

$$\dfrac{C_1 \vee A \qquad \dfrac{C_2 \vee \neg A \vee B \quad C_3 \vee \neg B}{C_2 \vee \neg A \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $A \succ B$, then the second proof does not fulfill the orderings restrictions.

## Avoiding Rotation Redundancy

*Conclusion:* In the presence of orderings restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representant in any class of of rotation-equivalent proofs.

# Lifting Lemma for $Res_S^{\succ}$

Lemma 2.39:

Let $C$ and $D$ be variable-disjoint clauses. If

$$\begin{array}{cc} C & D \\ \downarrow \sigma & \downarrow \rho \\ \hline C\sigma & D\rho \\ \hline \multicolumn{2}{c}{C'} \end{array} \qquad \text{[propositional inference in } Res_S^{\succ}\text{]}$$

and if $S(C\sigma) \simeq S(C)$, $S(D\rho) \simeq S(D)$ (that is, "corresponding" literals are selected), then there exists a substitution $\tau$ such that

$$\begin{array}{c} \dfrac{C \qquad D}{C''} \\ \downarrow \tau \\ C' = C''\tau \end{array} \qquad \text{[Inference in } Res_S^{\succ}\text{]}$$

# Lifting Lemma for $Res_S^{\succ}$

An analogous lifting lemma holds for factorization.

## Saturation of General Clause Sets

Corollary 2.40:

Let $N$ be a set of general clauses saturated under $Res_S^{\succ}$, i.e. $Res_S^{\succ}(N) \subseteq N$. Then there exists a selection function $S'$ such that $S|_N = S'|_N$ and $G_\Sigma(N)$ is also saturated, i.e.,

$$Res_{S'}^{\succ}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof:

We first define the selection function $S'$ such that $S'(C) = S(C)$ for all clauses $C \in G_\Sigma(N) \cap N$. For $C \in G_\Sigma(N) \setminus N$ we choose a fixed but arbitrary clause $D \in N$ with $C \in G_\Sigma(D)$ and define $S'(C)$ to be those occurrences of literals that are ground instances of the occurrences selected by $S$ in $D$. Then proceed as in the proof of Corollary 2.32 using the above lifting lemma.

# Soundness and Refutational Completeness

Theorem 2.41:
Let $\succ$ be an atom ordering and $S$ a selection function such that $Res_S^{\succ}(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N$$

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider first the propositional level: Construct a candidate model $I_N$ as for unrestricted resolution, except that clauses $C$ in $N$ that have selected literals are not productive, even when they are false in $I_C$ and when their maximal atom occurs only once and positively. The result for general clauses follows using Corollary 2.40.

# Craig-Interpolation

A theoretical application of ordered resolution is Craig-Interpolation:

Theorem 2.42 (Craig 57):
Let $F$ and $G$ be two propositional formulas such that $F \models G$. Then there exists a formula $H$ (called the interpolant for $F \models G$), such that $H$ contains only prop. variables occurring both in $F$ and in $G$, and such that $F \models H$ and $H \models G$.

# Craig-Interpolation

Proof:

Translate $F$ and $\neg G$ into CNF. let $N$ and $M$, resp., denote the resulting clause set. Choose an atom ordering $\succ$ for which the prop. variables that occur in $F$ but not in $G$ are maximal. Saturate $N$ into $N^*$ wrt. $Res_S^{\succ}$ with an empty selection function $S$ . Then saturate $N^* \cup M$ wrt. $Res_S^{\succ}$ to derive $\bot$. As $N^*$ is already saturated, due to the ordering restrictions only inferences need to be considered where premises, if they are from $N^*$, only contain symbols that also occur in $G$. The conjunction of these premises is an interpolant $H$. The theorem also holds for first-order formulas. For universal formulas the above proof can be easily extended. In the general case, a proof based on resolution technology is more complicated because of Skolemization.

# Redundancy

So far: local restrictions of the resolution inference rules using orderings and selection functions.

Is it also possible to delete clauses altogether?
Under which circumstances are clauses unnecessary?
(Conjecture: e. g., if they are tautologies or if they are subsumed by other clauses.)

Intuition: If a clause is guaranteed to be neither a minimal counterexample nor productive, then we do not need it.

## A Formal Notion of Redundancy

Let $N$ be a set of ground clauses and $C$ a ground clause (not necessarily in $N$). $C$ is called redundant w. r. t. $N$, if there exist $C_1, \dots, C_n \in N$, $n \geq 0$, such that $C_i \prec C$ and $C_1, \dots, C_n \models C$.

Redundancy for general clauses:
$C$ is called redundant w. r. t. $N$, if all ground instances $C\sigma$ of $C$ are redundant w. r. t. $G_\Sigma(N)$.

Intuition: Redundant clauses are neither minimal counterexamples nor productive.

Note: The same ordering $\succ$ is used for ordering restrictions and for redundancy (and for the completeness proof).

# Examples of Redundancy

Proposition 2.43:

- $C$ tautology (i.e., $\models C$) $\Rightarrow$ $C$ redundant w. r. t. any set $N$.
- $C\sigma \subset D$ $\Rightarrow$ $D$ redundant w. r. t. $N \cup \{C\}$
- $C\sigma \subseteq D$ $\Rightarrow$ $D \vee \overline{L}\sigma$ redundant w. r. t. $N \cup \{C \vee L,\ D\}$

(Under certain conditions one may also use non-strict subsumption, but this requires a slightly more complicated definition of redundancy.)

# Saturation up to Redundancy

$N$ is called saturated up to redundancy (wrt. $Res_S^{\succ}$)

$$:\Leftrightarrow Res_S^{\succ}(N \setminus Red(N)) \subseteq N \cup Red(N)$$

Theorem 2.44:

Let $N$ be saturated up to redundancy. Then

$$N \models \bot \Leftrightarrow \bot \in N$$

# Saturation up to Redundancy

Proof (Sketch):

(i) Ground case:

- consider the construction of the candidate model $I_N^{\succ}$ for $Res_S^{\succ}$
- redundant clauses are not productive
- redundant clauses in $N$ are not minimal counterexamples for $I_N^{\succ}$

The premises of "essential" inferences are either minimal counterexamples or productive.

(ii) Lifting: no additional problems over the proof of Theorem 2.41.

# Monotonicity Properties of Redundancy

Theorem 2.45:

(i) $N \subseteq M \Rightarrow \mathit{Red}(N) \subseteq \mathit{Red}(M)$

(ii) $M \subseteq \mathit{Red}(N) \Rightarrow \mathit{Red}(N) \subseteq \mathit{Red}(N \setminus M)$

Proof: Exercise.

We conclude that redundancy is preserved when, during a theorem proving process, one adds (derives) new clauses or deletes redundant clauses.

# A Resolution Prover

So far: static view on completeness of resolution:

Saturated sets are inconsistent if and only if they contain $\perp$.

We will now consider a dynamic view:

How can we get saturated sets in practice?

The theorems 2.44 and 2.45 are the basis for the completeness proof of our prover $RP$.

## Rules for Simplifications and Deletion

We want to employ the following rules for simplification of prover states $N$:

- Deletion of tautologies

$$N \cup \{C \vee A \vee \neg A\} \;\triangleright\; N$$

- Deletion of subsumed clauses

$$N \cup \{C, D\} \;\triangleright\; N \cup \{C\}$$

  if $C\sigma \subseteq D$ ($C$ subsumes $D$).

- Reduction (also called subsumption resolution)

$$N \cup \{C \vee L,\;\; D \vee C\sigma \vee \overline{L}\sigma\} \;\triangleright\; N \cup \{C \vee L, D \vee C\sigma\}$$

# Resolution Prover *RP*

**3 clause sets:** N(ew) containing new resolvents

P(rocessed) containing simplified resolvents

clauses get into O(ld) once their inferences have been computed

**Strategy:** Inferences will only be computed when there are no possibilities for simplification

## Transition Rules for *RP* (I)

Tautology elimination

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \triangleright \quad \boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if $C$ is a tautology

Forward subsumption

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \triangleright \quad \boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if some $D \in \boldsymbol{P} \cup \boldsymbol{O}$ subsumes $C$

Backward subsumption

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \cup \{D\} \mid \boldsymbol{O} \quad \triangleright \quad \boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \cup \{D\} \quad \triangleright \quad \boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if $C$ strictly subsumes $D$

# Transition Rules for *RP* (II)

Forward reduction

$$\boldsymbol{N} \cup \{C \vee L\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if there exists $D \vee L' \in \boldsymbol{P} \cup \boldsymbol{O}$

such that $\overline{L} = L'\sigma$ and $D\sigma \subseteq C$

Backward reduction

$$\boldsymbol{N} \mid \boldsymbol{P} \cup \{C \vee L\} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O}$$

$$\boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O} \cup \{C \vee L\} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O}$$

if there exists $D \vee L' \in \boldsymbol{N}$

such that $\overline{L} = L'\sigma$ and $D\sigma \subseteq C$

# Transition Rules for *RP* (III)

Clause processing

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O}$$

Inference computation

$$\emptyset \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O} \cup \{C\},$$

$$\text{with } \boldsymbol{N} = \mathit{Res}_S^{\succ}(\boldsymbol{O} \cup \{C\})$$

# Soundness and Completeness

Theorem 2.46:

$$N \models \bot \quad \Leftrightarrow \quad N \mid \emptyset \mid \emptyset \quad \overset{*}{\rhd} \quad N' \cup \{\bot\} \mid \_ \mid \_$$

Proof in

L. Bachmair, H. Ganzinger: Resolution Theorem Proving
(on H. Ganzinger's Web page under Publications/Journals;
appeared in the Handbook on Automated Theorem Proving,
2001)