Recursive Path Orderings

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $>$ be a strict partial ordering ("precedence") on Ω. The lexicographic path ordering $>_{\mathrm{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $>$ is defined by: $s >_{\mathrm{lpo}} t$ iff

(1) $t \in \mathrm{Var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

   (a) $s_i \geq_{\mathrm{lpo}} t$ for some i, or

   (b) $f > g$ and $s >_{\mathrm{lpo}} t_j$ for all j, or

   (c) $f = g$, $s >_{\mathrm{lpo}} t_j$ for all j, and
   $(s_1, \dots, s_m)\ (>_{\mathrm{lpo}})_{\mathrm{lex}}\ (t_1, \dots, t_n)$.

Recursive Path Orderings

There are several possibilities to compare subterms in (2)(c):

- compare list of subterms lexicographically left-to-right ("lexicographic path ordering (lpo)", Kamin and Lévy)
- compare list of subterms lexicographically right-to-left (or according to some permutation π)
- compare multiset of subterms using the multiset extension ("multiset path ordering (mpo)", Dershowitz)
- to each function symbol f/n associate a status $\in \{mul\} \cup \{\, lex_\pi \mid \pi : \{1,\ldots,n\} \to \{1,\ldots,n\}\,\}$ and compare according to that status ("recursive path ordering (rpo) with status")

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature,
let $>$ be a strict partial ordering ("precedence") on Ω,
let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a weight function,
such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$;
$w(c) \geq w_0$ for all constants $c/0 \in \Omega$.

If $w(f) = 0$ for some $f/1 \in \Omega$, then $f \geq g$ for all $g \in \Omega$.

w can be extended to terms as follows:

$$w(t) = \sum_{x \in \mathrm{Var}(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t).$$

The Knuth-Bendix Ordering

The Knuth-Bendix ordering $>_{\mathrm{kbo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $>$ and w is defined by: $s >_{\mathrm{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables x, $w(s) = w(t)$, and

  (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or

  (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f > g$, or

  (c) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m) \; (>_{\mathrm{kbo}})_{\mathrm{lex}} \; (t_1, \ldots, t_m)$.

The Knuth-Bendix Ordering

Theorem:

The Knuth-Bendix ordering induced by $>$ and w is a simplification ordering on $\mathrm{T}_\Sigma(X)$.

Proof:

Baader and Nipkow, pages 125–129.

6 Knuth-Bendix Completion

Knuth-Bendix Completion

Completion:

Goal: Given a set E of equations, transform E into an equivalent convergent set R of rewrite rules.

How to ensure termination?

Fix a reduction ordering $>$ and construct R in such a way that $\rightarrow_R \subseteq >$ (i. e., $l > r$ for every $l \rightarrow r \in R$).

How to ensure confluence?

Check that all critical pairs are joinable.

Knuth-Bendix Completion: Inference Rules

The completion procedure is presented as a set of inference rules working on a set of equations E and a set of rules R:
$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$

At the beginning, $E = E_0$ is the input set and $R = R_0$ is empty. At the end, E should be empty; then R is the result.

For each step $E, R \vdash E', R'$, the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Knuth-Bendix Completion: Inference Rules

Notations:

The formula $s \dot{\approx} t$ denotes either $s \approx t$ or $t \approx s$.

$\mathrm{CP}(R)$ denotes the set of all critical pairs between rules in R.

Knuth-Bendix Completion: Inference Rules

Orient:

$$\frac{E \cup \{s \mathrel{\dot{\approx}} t\}, \quad R}{E, \quad R \cup \{s \to t\}} \qquad \text{if } s > t$$

Note: There are equations $s \approx t$ that cannot be oriented,
i. e., neither $s > t$ nor $t > s$.

Knuth-Bendix Completion: Inference Rules

Trivial equations cannot be oriented – but we don't need them anyway:

Delete:

$$\frac{E \cup \{s \approx s\}, \quad R}{E, \quad R}$$

Knuth-Bendix Completion: Inference Rules

Critical pairs between rules in R are turned into additional equations:

Deduce:

$$\frac{E, \quad R}{E \cup \{s \approx t\}, \quad R} \qquad \text{if } \langle s, t \rangle \in \mathrm{CP}(R).$$

Note: If $\langle s, t \rangle \in R$ then $s \leftarrow_R u \rightarrow_R t$ and hence $R \models s \approx t$.

Knuth-Bendix Completion: Inference Rules

The following inference rules are not absolutely necessary, but very useful (e.g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

Simplify-Eq:

$$\frac{E \cup \{s \dot{\approx} t\}, \quad R}{E \cup \{u \approx t\}, \quad R} \qquad \text{if } s \rightarrow_R u.$$

Knuth-Bendix Completion: Inference Rules

Simplification of the right-hand side of a rule is unproblematic.

R-Simplify-Rule:

$$\frac{E, \quad R \cup \{s \to t\}}{E, \quad R \cup \{s \to u\}} \qquad \text{if } t \to_R u.$$

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

L-Simplify-Rule:

$$\frac{E, \quad R \cup \{s \to t\}}{E \cup \{u \approx t\}, \quad R} \qquad \begin{array}{l}\text{if } s \to_R u \text{ using a rule } l \to r \in R \\ \text{such that } s \sqsupset l \text{ (see next slide).}\end{array}$$

Knuth-Bendix Completion: Inference Rules

For technical reasons, the lhs of $s \to t$ may only be simplified using a rule $l \to r$, if $l \to r$ *cannot* be simplified using $s \to t$, that is, if $s \sqsupset l$, where the encompassment quasi-ordering $\mathrel{\underset{\sim}{\sqsupset}}$ is defined by

$$s \mathrel{\underset{\sim}{\sqsupset}} l \ \text{ if } \ s/p = l\sigma \text{ for some } p \text{ and } \sigma$$

and $\sqsupset = \mathrel{\underset{\sim}{\sqsupset}} \setminus \mathrel{\underset{\sim}{\sqsubset}}$ is the strict part of $\mathrel{\underset{\sim}{\sqsupset}}$.

Lemma:
$\sqsupset$ is a well-founded strict partial ordering.

Knuth-Bendix Completion: Inference Rules

Lemma:
If $E, R \vdash E', R'$, then $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Lemma:
If $E, R \vdash E', R'$ and $\rightarrow_R \subseteq >$, then $\rightarrow_{R'} \subseteq >$.

Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set E of equations, different things can happen:

(1) We reach a state where no more inference rules are applicable and E is not empty.
$\Rightarrow$ Failure (try again with another ordering?)

(2) We reach a state where E is empty and all critical pairs between the rules in the current R have been checked.

(3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.

Knuth-Bendix Completion: Correctness Proof

A (finite or infinite sequence) $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ with $R_0 = \emptyset$ is called a run of the completion procedure with input E_0 and $>$.

For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

The sets of persistent equations or rules of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n, then $E_* = E_n$ and $R_* = R_n$.

Knuth-Bendix Completion: Correctness Proof

A run is called fair, if $CP(R_*) \subseteq E_\infty$
(i. e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and E_* is empty, then R_* is convergent and equivalent to E_0.

In particular: If a run is fair and E_* is empty, then $\approx_{E_0} = \approx_{E_\infty \cup R_\infty} = \leftrightarrow_{E_\infty \cup R_\infty} = \downarrow_{R_*}$.

Knuth-Bendix Completion: Correctness Proof

General assumptions from now on:

$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ is a fair run.

R_0 and E_* are empty.

Knuth-Bendix Completion: Correctness Proof

A proof of $s \approx t$ in $E_\infty \cup R_\infty$ is a finite sequence $(s_0, \ldots, s_n)$ such that $s = s_0$, $t = s_n$, and for all $i \in \{1, \ldots, n\}$:

(1) $s_{i-1} \leftrightarrow_{E_\infty} s_i$, or

(2) $s_{i-1} \rightarrow_{R_\infty} s_i$, or

(3) $s_{i-1} \leftarrow_{R_\infty} s_i$.

The pairs (s_{i-1}, s_i) are called proof steps.

A proof is called a rewrite proof in R_*,
if there is a $k \in \{0, \ldots, n\}$ such that $s_{i-1} \rightarrow_{R_*} s_i$ for $1 \leq i \leq k$
and $s_{i-1} \leftarrow_{R_*} s_i$ for $k + 1 \leq i \leq n$

Knuth-Bendix Completion: Correctness Proof

Idea (Bachmair, Dershowitz, Hsiang):

Define a well-founded ordering on proofs, such that for every proof that is not a rewrite proof in R_* there is an equivalent smaller proof.

Consequence: For every proof there is an equivalent rewrite proof in R_*.

Knuth-Bendix Completion: Correctness Proof

We associate a cost $c(s_{i-1}, s_i)$ with every proof step as follows:

(1) If $s_{i-1} \leftrightarrow_{E_\infty} s_i$, then $c(s_{i-1}, s_i) = (\{s_{i-1}, s_i\}, -, -)$, where the first component is a multiset of terms and $-$ denotes an arbitrary (irrelevant) term.

(2) If $s_{i-1} \rightarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_{i-1}\}, l, s_i)$.

(3) If $s_{i-1} \leftarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_i\}, l, s_{i-1})$.

Proof steps are compared using the lexicographic combination of the multiset extension of reduction ordering $>$, the encompassment ordering $\sqsupset$, and the reduction ordering $>$.

Knuth-Bendix Completion: Correctness Proof

The cost $c(P)$ of a proof P is the multiset of the costs of its proof steps.

The proof ordering $>_C$ compares the costs of proofs using the multiset extension of the proof step ordering.

Lemma:
$>_C$ is a well-founded ordering.

Knuth-Bendix Completion: Correctness Proof

Lemma:

Let P be a proof in $E_\infty \cup R_\infty$. If P is not a rewrite proof in R_*, then there exists an equivalent proof P' in $E_\infty \cup R_\infty$ such that $P >_C P'$.

Proof:

If P is not a rewrite proof in R_*, then it contains

(a) a proof step that is in E_∞, or

(b) a proof step that is in $R_\infty \setminus R_*$, or

(c) a subproof $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$ (peak).

We show that in all three cases the proof step or subproof can be replaced by a smaller subproof:

Knuth-Bendix Completion: Correctness Proof

Case (a): A proof step using an equation $s \doteq t$ is in E_∞.
This equation must be deleted during the run.

If $s \doteq t$ is deleted using *Orient*:

$$\ldots s_{i-1} \leftrightarrow_{E_\infty} s_i \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \rightarrow_{R_\infty} s_i \ldots$$

If $s \doteq t$ is deleted using *Delete*:

$$\ldots s_{i-1} \leftrightarrow_{E_\infty} s_{i-1} \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \ldots$$

If $s \doteq t$ is deleted using *Simplify-Eq*:

$$\ldots s_{i-1} \leftrightarrow_{E_\infty} s_i \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \ldots$$

Knuth-Bendix Completion: Correctness Proof

Case (b): A proof step using a rule $s \rightarrow t$ is in $R_\infty \setminus R_*$.
This rule must be deleted during the run.

If $s \rightarrow t$ is deleted using *R-Simplify-Rule*:

$$\ldots s_{i-1} \rightarrow_{R_\infty} s_i \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \rightarrow_{R_\infty} s' \leftarrow_{R_\infty} s_i \ldots$$

If $s \rightarrow t$ is deleted using *L-Simplify-Rule*:

$$\ldots s_{i-1} \rightarrow_{R_\infty} s_i \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \ldots$$

Knuth-Bendix Completion: Correctness Proof

Case (c): A subproof has the form $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$.

If there is no overlap or a non-critical overlap:

$$\ldots s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1} \ldots \implies \ldots s_{i-1} \rightarrow^*_{R_*} s' \leftarrow^*_{R_*} s_{i+1} \ldots$$

If there is a critical pair that has been added using *Deduce*:

$$\ldots s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1} \ldots \implies \qquad \ldots s_{i-1} \leftrightarrow_{E_\infty} s_i \ldots$$

In all cases, checking that the replacement subproof is smaller than the replaced subproof is routine.

Knuth-Bendix Completion: Correctness Proof

Theorem:
Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ be a fair run and let R_0 and E_* be empty. Then

(1) every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in R_*,

(2) R_* is equivalent to E_0, and

(3) R_* is convergent.

Knuth-Bendix Completion: Correctness Proof

Proof:

(1) By well-founded induction on $>_C$ using the previous lemma.

(2) Clearly $\approx_{E_\infty \cup R_\infty} = \approx_{E_0}$.
Since $R_* \subseteq R_\infty$, we get $\approx_{R_*} \subseteq \approx_{E_\infty \cup R_\infty}$.
On the other hand, by (1), $\approx_{E_\infty \cup R_\infty} \subseteq \approx_{R_*}$.

(3) Since $\rightarrow_{R_*} \subseteq >$, R_* is terminating.
By (1), R_* is confluent.