Counting and Finding Homomorphisms is Universal for Parameterized Complexity Theory

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Finding Graph Homomorphisms

**Graph Homomorphism**
Mapping from graph $H$ to $G$ that preserves edges;
Write $\text{Hom}(H \rightarrow G)$ for the set of all graph hom’s from $H$ to $G$. 

$H$

$G$
Finding Graph Homomorphisms

**Graph Homomorphism**

Mapping from graph $H$ to $G$ that preserves edges; Write $\text{Hom}(H \to G)$ for the set of all graph hom’s from $H$ to $G$. 

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Finding Graph Homomorphisms

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Write Hom($H \rightarrow G$) for the set of all graph hom’s from $H$ to $G$. 

$\Phi = \text{bipartite } H \mid V(H) = 4 \mid G$
Finding Graph Homomorphisms

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Write $\text{Hom}(H \to G)$ for the set of all graph hom’s from $H$ to $G$.

$\Phi = \text{bipartite } H | V(H) | = 4$

$\#\text{Hom}(H \to G) = 14$

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Counting Homomorphisms is Universal for Parameterized Complexity Theory
Finding Graph Homomorphisms

**Graph Homomorphism**

Mapping from graph $H$ to $G$ that preserves edges;
Write $\text{Hom}(H \to G)$ for the set of all graph hom’s from $H$ to $G$. 

\[
\Phi = \text{bipartite } H | V(H) | = 4 \rightarrow G
\]
Finding Graph Homomorphisms

**Graph Homomorphism**

Mapping from graph $H$ to $G$ that preserves edges;
Write $\text{Hom}(H \to G)$ for the set of all graph hom’s from $H$ to $G$.

No homomorphisms from $H$ to $G$. 
Finding Graph Homomorphisms

\[ \text{Hom}(H \to G) \]

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph homomorphism from \( H \) to \( G \).
Finding Graph Homomorphisms

**Hom**($\mathcal{H} \to \mathcal{G}$)

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph hom from $H$ to $G$. 

**Graph classes**
Finding Graph Homomorphisms

\( \text{Hom}(\mathcal{H} \to G) \)

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph hom from \( H \) to \( G \).

Graph classes

- All Graphs (\( \top \))
- All Bipartite Graphs
- All Cliques

Counting Homomorphisms is Universal for Parameterized Complexity Theory
Finding Graph Homomorphisms

\( \text{Hom}(H \rightarrow G) \)

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph homomorphism from \( H \) to \( G \).

Graph classes

- All Graphs (\( \top \))
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Counting Homomorphisms is Universal for Parameterized Complexity Theory
Known Results

\( \text{Hom}(\mathcal{H} \to \mathcal{G}) \)

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph hom from \( H \) to \( G \).
Known Results

$$\text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$$
Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph hom from $H$ to $G$.

NP-complete

$$\text{Hom}(\top \rightarrow \top)$$
Known Results

$\text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph hom from $H$ to $G$.

NP-complete

$\text{Hom}(\top \rightarrow \top)$

$3$-COLORABLE
Known Results

\textbf{\textsc{Hom}}(\mathcal{H} \rightarrow \mathcal{G})

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph hom from $H$ to $G$.

\textbf{NP-complete}

\textbf{\textsc{Hom}}(\top \rightarrow \{\triangle\})

3-COLORABLE

Counting Homomorphisms is Universal for Parameterized Complexity Theory

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# Known Results

\[ \text{Hom}(H \to G) \]

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph hom from \( H \) to \( G \).

Are there fast algorithms for special cases of \( \text{Hom}(\top \to \top) \)?
Known Results

**$\text{Hom}(\mathcal{H} \to \mathcal{G})$**

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph hom from $H$ to $G$.

What makes $\text{Hom}(\top \to \top)$ hard?
Known Results

\[ \text{Hom}(\mathcal{H} \to \mathcal{G}) \]

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph hom from \( H \) to \( G \).

<table>
<thead>
<tr>
<th>( \text{Hom}(\top \to \mathcal{G}) )</th>
<th>( \mathcal{G} ) contains only bipartite graphs</th>
<th>( \mathcal{G} ) contains a non-bipartite graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>poly-time solvable</td>
<td>( \mathcal{G} ) contains only bipartite graphs</td>
<td>( \mathcal{G} ) contains a non-bipartite graph</td>
</tr>
<tr>
<td>[\text{[Hell, Nešetřil '90]}]</td>
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</tr>
</tbody>
</table>

\[ \text{Hom}(\mathcal{H} \to \mathcal{G}) \] is poly-time solvable for \( \mathcal{H} = \top \), but NP-complete for \( \mathcal{H} \neq \top \).
Known Results

$\#\text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count all graph homomorphisms from $H$ to $G$.

<table>
<thead>
<tr>
<th>poly-time solvable</th>
<th>$#P$-complete</th>
</tr>
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<tbody>
<tr>
<td>$#\text{Hom}(\top \rightarrow \mathcal{G})$</td>
<td>(explicit criterion exists)</td>
</tr>
<tr>
<td>[Dyer, Greenhill ’00]</td>
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</table>
Known Results

\textbf{Hom}(\mathcal{H} \rightarrow \mathcal{G})

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph hom from \( H \) to \( G \).

What about \textit{the other side}, \( \text{Hom}(\mathcal{H} \rightarrow \top) \)?
### Known Results

**$\text{Hom}(H \rightarrow G)$**

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph homomorphism from $H$ to $G$.

What about the other side, $\text{Hom}(\mathcal{H} \rightarrow \top)$?
## Known Results

| **\( \text{Hom}(\mathcal{H} \to \mathcal{G}) \)** | Parameter: \(|V(H)|\) |
|-----------------------------------------------|------------------|
| Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), check if there is a graph hom from \( H \) to \( G \). | |

<table>
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<tr>
<th>****</th>
<th><strong>FPT</strong></th>
<th><strong>W[1]-hard</strong></th>
</tr>
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<td>( \text{Hom}(\mathcal{H} \to \top) )</td>
<td>( (f(</td>
<td>V(H)</td>
</tr>
</tbody>
</table>

- “\( \mathcal{H} \) contains only graphs with small treewidth”
  - \[\text{Grohe '03}\]
- “\( \mathcal{H} \) contains graphs with arbitrary large tw”
  - \[\text{Grohe '03}\]
### Known Results

**Parameter:** $|V(H)|$

| $\#\text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$ | FPT $(f(|V(H)|) \cdot \text{poly}(|V(G)|)$ time) | $\#W[1] \text{-hard}$ (not faster than $\#\text{K-CLIQUE}$) |
|-------------------------------------------------|-----------------------------------------------|--------------------------------------------------|
| Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count all graph homomorphisms from $H$ to $G$. | “$\mathcal{H}$ contains only graphs with small treewidth” | “$\mathcal{H}$ contains a graph with large treewidth” |
| $\#\text{Hom}(\mathcal{H} \rightarrow T)$ | [Dalmau, Jonsson ’04] | [Dalmau, Jonsson ’04] |
Known Results

\[ \text{\#Hom}(\mathcal{H} \to \mathcal{G}) \]

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count all graph homomorphisms from $H$ to $G$.

Complexity dichotomies when restricting either $\mathcal{G}$ or $\mathcal{H}$. 
Known Results

$\#\text{Hom}(\mathcal{H} \to \mathcal{G})$

Parameter: $|V(H)|$

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count all graph homomorphisms from $H$ to $G$.

Complexity dichotomies when restricting either $\mathcal{G}$ or $\mathcal{H}$.

What if we restrict both sides?
Known Results

\[
\#\text{Hom}(\mathcal{H} \to \mathcal{G})
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Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), count all graph homomorphisms from \( H \) to \( G \).

Parameter: \(|V(H)|\)

Complexity dichotomies when restricting either \( \mathcal{G} \) or \( \mathcal{H} \).

What if we restrict both sides?

This talk.
Main Result

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count all graph homomorphisms from $H$ to $G$.

Theorem

For any problem $P$ in $\#W[1]$ (or $W[1]$), there are graph classes $\mathcal{H}_P$ and $\mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$ (or $\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$).
Main Result

\[ \text{Parameter: } |V(H)| \]

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), count all graph homomorphisms from \( H \) to \( G \).

Theorem

For any problem \( P \) in \( \#W[1] \) (or \( W[1] \)), there are graph classes \( \mathcal{H}_P \) and \( \mathcal{G}_P \) such that \( P \) is equivalent to \( \#\text{HOM}(\mathcal{H}_P \to \mathcal{G}_P) \) (or \( \text{HOM}(\mathcal{H}_P \to \mathcal{G}_P) \)).

- Cannot hope for clear categorization into \( \text{FPT}/W[1] \)-hard for all pairs \((\mathcal{H}, \mathcal{G})\) (think of Ladner’s Theorem)
Proof Ideas

\[ \text{#Hom}(\mathcal{H} \rightarrow \mathcal{G}) \]

Parameter: \( |V(H)| \)

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), count all graph homomorphisms from \( H \) to \( G \).

Theorem

For any problem \( P \) in \( \text{#W}[1] \) (or \( W[1] \)), there are graph classes \( \mathcal{H}_P \) and \( \mathcal{G}_P \) such that \( P \) is equivalent to \( \text{#Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P) \) (or \( \text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P) \)).
Proof Ideas

\( \text{Parameter: } |V(H)| \)

\( \#\text{HOM}(H \rightarrow G) \)

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), count all graph homomorphisms from \( H \) to \( G \).

Theorem

For any problem \( P \) in \( \#W[1] \) (or \( W[1] \)), there are graph classes \( \mathcal{H}_P \) and \( \mathcal{G}_P \) such that \( P \) is equivalent to \( \#\text{HOM}(\mathcal{H}_P \rightarrow \mathcal{G}_P) \) (or \( \text{HOM}(\mathcal{H}_P \rightarrow \mathcal{G}_P) \)).

Recall: \( \#\text{HOM}(\mathcal{H} \rightarrow T) \) is \( \#W[1] \)-hard if \( \mathcal{H} \) has “unbounded treewidth” [DalJon’04]
Proof Ideas

**#Hom(H \rightarrow G)**

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count the number of graph hom's from $H$ to $G$.

**Theorem**

For any $P$ in #W[1], there are $\mathcal{H}_P, \mathcal{G}_P$ such that $P$ is equivalent to $\#Hom(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.
Proof Ideas

\[ \text{Parameter: } |V(H)| \]

Given graphs \( H \in \mathcal{H} \) and \( G \in \mathcal{G} \), count the number of graph homomorphisms from \( H \) to \( G \).

Theorem

For any \( P \) in \#W[1], there are \( \mathcal{H}_P, \mathcal{G}_P \) such that \( P \) is equivalent to \#\text{OM}(\mathcal{H}_P \rightarrow \mathcal{G}_P).

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**Proof Ideas**

\[ \text{Parameter: } |V(H)| \]

**Theorem**

For any \( P \) in \( \#W[1] \), there are \( \mathcal{H}_P, \mathcal{G}_P \) such that \( P \) is equivalent to \( \#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P) \).

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Counting Homomorphisms is Universal for Parameterized Complexity Theory
Proof Ideas

**Theorem**

For any $P$ in $\#W[1]$, there are $\mathcal{H}_P$, $\mathcal{G}_P$ such that $P$ is equivalent to $\#\text{HOM}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.
Proof Ideas

**Theorem**

For any $P$ in $\#W[1]$, there are $\mathcal{H}_P, G_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \rightarrow G_P)$.
Proof Ideas

Theorem
For any $P$ in $\#W[1]$, there are $H_P, G_P$ such that $P$ is equivalent to $\#\text{HOM}(H_P \rightarrow G_P)$.

Instance $J$
of $P$

Problem $P$

$\#\text{HOM}(H \rightarrow T)$

Graphs $H_J, G_J$
**Theorem**

For any $P$ in $\#W[1]$, there are $\mathcal{H}_P$, $\mathcal{G}_P$ such that $P$ is equivalent to $\#\text{HOM}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.

**Proof Ideas**

```
Instance J of P       Problem P          #HOM(\mathcal{H} \rightarrow \top)       Graphs H_J, G_J
```

Approach:

\[
\mathcal{H}_P := \{ H_J \mid \text{instance } J \text{ of } P \}
\]

\[
\mathcal{G}_P := \{ G_J \mid \text{instance } J \text{ of } P \}
\]
Proof Ideas

**Theorem**
For any $P$ in $\#W[1]$, there are $\mathcal{H}_P, \mathcal{G}_P$ such that $P$ is equivalent to $\#\text{HOM}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.

Approach:
- $\mathcal{H}_P := \{H_J \mid \text{instance } J \text{ of } P\}$
- $\mathcal{G}_P := \{G_J \mid \text{instance } J \text{ of } P\}$

$P \preceq \#\text{HOM}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$ \(\checkmark\)
Proof Ideas

**Theorem**

For any $P$ in $\#W[1]$, there are $\mathcal{H}_P, \mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.

**Approach:**

$\mathcal{H}_P := \{ H_J | \text{instance } J \text{ of } P \}$

$\mathcal{G}_P := \{ G_J | \text{instance } J \text{ of } P \}$

$P \preceq \#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$

$\#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P) \preceq P$
Proof Ideas

**Theorem**

For any $P$ in $\#W[1]$, there are $H_P, G_P$ such that $P$ is equivalent to $\#HOM(H_P \to G_P)$.

**Approach:**

$H_P := \{H_J \mid \text{instance } J \text{ of } P\}$

$G_P := \{G_J \mid \text{instance } J \text{ of } P\}$

$P \preceq \#HOM(H_P \to G_P)$

$\#HOM(H_P \to G_P) \not\preceq P$

How do we obtain instance $J$ from $(H_J, G_J)$?
Proof Ideas

**Theorem**

For any $P$ in $\#W[1]$, there are $H_P, G_P$ such that $P$ is equivalent to $\#\text{HOM}(H_P \rightarrow G_P)$.

Instance $J$ of $P$ encode $\langle J \rangle$

Problem $P \rightarrow \#\text{HOM}(H \rightarrow \top)$

Graphs $H_J, G_J$

Graph $\langle J \rangle$
Proof Ideas

**Theorem**
For any $P$ in $\mathbf{W}[1]$, there are $\mathcal{H}_P, \mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$.

Approach:

$$
\mathcal{H}_P := \{H_J \mid \text{instance } J \text{ of } P\} \\
\mathcal{G}_P := \{G_J \cup \langle J \rangle \mid \text{instance } J \text{ of } P\}
$$
Proof Ideas

**Theorem**
For any $P$ in $\#W[1]$, there are $H_P, G_P$ such that $P$ is equivalent to $\#\text{HOM}(H_P \rightarrow G_P)$.

Approach:
- $H_P := \{H_J \mid \text{instance } J \text{ of } P\}$
- $G_P := \{G_J \cup \langle J \rangle \mid \text{instance } J \text{ of } P\}$

$P \preccurlyeq \#\text{HOM}(H_P \rightarrow G_P)$

(ensure $\text{Hom}(H_J \rightarrow \langle J \rangle) = 0$)
Proof Ideas

**Theorem**

For any $P$ in $\#W[1]$, there are $\mathcal{H}_P$, $\mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.

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$\mathcal{H}_P := \{H_J \mid \text{instance } J \text{ of } P\}$

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$P \preccurlyeq \#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$

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Proof Ideas

**Theorem**
For any $P$ in $\#W[1]$, there are $\mathcal{H}_P, G_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \to G_P)$.

Approach:

$$\mathcal{H}_P := \{ H_J \mid \text{instance } J \text{ of } P \}$$

$$G_P := \{ G_J \cup \langle J \rangle \mid \text{instance } J \text{ of } P \}$$

$$P \preceq \#\text{Hom}(\mathcal{H}_P \to G_P)$$

(ensure $\text{Hom}(H_J \to \langle J \rangle) = 0$)

$$\#\text{Hom}(\mathcal{H}_P \to G_P) \preceq P$$

How do we handle malformed input $(H_J, G_L)$?
Proof Ideas

**Theorem**
For any $P$ in #$W[1]$, there are $\mathcal{H}_P$, $G_P$ such that $P$ is equivalent to #$\text{OM}(\mathcal{H}_P \to G_P)$.

**Approach:**
\[
\begin{align*}
\mathcal{H}_P & := \{H_J \mid \text{instance } J \text{ of } P\} \\
G_P & := \{G_J \cup \langle J \rangle \mid \text{instance } J \text{ of } P\}
\end{align*}
\]

$P \leq \#\text{OM}(\mathcal{H}_P \to G_P) \checkmark$
(ensure $\text{Hom}(H_J \to \langle J \rangle) = 0$)

$\#\text{OM}(\mathcal{H}_P \to G_P) \not\leq P$
How do we ensure $\text{Hom}(H_J \to G_L \cup \langle L \rangle) = 0$?
Theorem
For any \( P \) in \(#W[1]#\), there are \( \mathcal{H}_P, \mathcal{G}_P \) such that \( P \) is equivalent to \( \# \text{HOM}(\mathcal{H}_P \to \mathcal{G}_P) \).

\[
P \preccurlyeq \# \text{HOM}(\mathcal{H}_P \to \mathcal{G}_P) \quad \# \text{HOM}(\mathcal{H}_P \to \mathcal{G}_P) \preccurlyeq P
\]

Can solve instance \( J \) with \( (H_J, G_J \cup \langle J \rangle) \) by computing \( \# \text{Hom}(H_J \to G_J \cup \langle J \rangle) \) (ensuring \( \# \text{Hom}(H_J \to \langle J \rangle) = 0 \))

Can extract instance \( J \) from pair \( (H_J, G_J \cup \langle J \rangle) \)

How do we ensure \( \# \text{Hom}(H_J \to G_L \cup \langle L \rangle) = 0 \)?
Theorem

For any $P$ in $\#W[1]$, there are $\mathcal{H}_P, \mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$.

$P \preccurlyeq \#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$

Can solve instance $J$ with $(H_J, G_J \cup \langle J \rangle)$

by computing $\#\text{Hom}(H_J \rightarrow G_J \cup \langle J \rangle)$

(ensuring $\#\text{Hom}(H_J \rightarrow \langle J \rangle) = 0$)

Can extract instance $J$ from pair $(H_J, G_J \cup \langle J \rangle)$

How do we ensure $\#\text{Hom}(H_J \rightarrow G_L \cup \langle L \rangle) = 0$?

Instance $J$ of $P$  \hspace{1cm}  Problem $P$  \hspace{1cm}  $\#\text{Hom}(\mathcal{H} \rightarrow \mathcal{T})$  \hspace{1cm}  Graphs $H_J, G_J$
Theorem
For any $P$ in $\#W[1]$, there are $H_P$, $G_P$ such that $P$ is equivalent to $\#\text{HOM}(H_P \rightarrow G_P)$.

$P \preceq \#\text{HOM}(H_P \rightarrow G_P)$

Can solve instance $J$ with $(H_J, G_J \cup \langle J \rangle)$
by computing $\#\text{Hom}(H_J \rightarrow G_J \cup \langle J \rangle)$
(ensuring $\#\text{Hom}(H_J \rightarrow \langle J \rangle) = 0$)

$\#\text{HOM}(H_P \rightarrow G_P) \preceq P$

Can extract instance $J$ from pair $(H_J, G_J \cup \langle J \rangle)$
How do we ensure $\#\text{Hom}(H_J \rightarrow G_L \cup \langle L \rangle) = 0$?
Theorem

For any $P$ in $\#W[1]$, there are $H_P, G_P$ such that $P$ is equivalent to $\#\text{Hom}(H_P \rightarrow G_P)$.

\[
P \preceq \#\text{Hom}(H_P \rightarrow G_P) \quad \text{and} \quad \#\text{Hom}(H_P \rightarrow G_P) \preceq P
\]

Can solve instance $J$ with $(H_J, G_J \cup \langle J \rangle)$ by computing $\#\text{Hom}(H_J \rightarrow G_J \cup \langle J \rangle)$ (ensuring $\#\text{Hom}(H_J \rightarrow \langle J \rangle) = 0$)

Can extract instance $J$ from pair $(H_J, G_J \cup \langle J \rangle)$

How do we ensure $\#\text{Hom}(H_J \rightarrow G_L \cup \langle L \rangle) = 0$?

No homomorphisms
\[ \text{Hom}(H_J \rightarrow G_J) \]

\[ \text{Aut}(K(2\kappa(J) + 3)) \]

\[ \tilde{H}_J \]

\[ \hat{G}_J \]

\[ \langle J, H_J \rangle \]

\[ K(2\kappa(J) + 3) \]
Main Result

$\#\text{Hom}(\mathcal{H} \to \mathcal{G})$

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, count all graph homomorphisms from $H$ to $G$.

Theorem

For any problem $P$ in $\#W[1]$ (or $W[1]$), there are graph classes $\mathcal{H}_P$ and $\mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$ (or $\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$).

- Cannot hope for clear categorization into FPT/$W[1]$-hard for all pairs $(\mathcal{H}, \mathcal{G})$

$\Rightarrow$ Need to look at specific pairs of graph classes
Main Result

**Theorem**

For any problem $P$ in $\#W[1]$ (or $W[1]$), there are graph classes $\mathcal{H}_P$ and $\mathcal{G}_P$ such that $P$ is equivalent to $\#\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$ (or $\text{Hom}(\mathcal{H}_P \rightarrow \mathcal{G}_P)$).

- Cannot hope for clear categorization into FPT/$W[1]$-hard for all pairs $(\mathcal{H}, \mathcal{G})$
- Need to look at specific pairs of graph classes
Open Problems

- Can we find a “hierarchy” of homomorphism problems?
  \[ \text{#HOM}(\mathcal{H}_1 \to \mathcal{G}_1) \leq \text{#HOM}(\mathcal{H}_2 \to \mathcal{G}_2) \leq \cdots \leq \text{#HOM}(\top \to \top) \]

- (Grunt work?) Obtain algorithms/hardness for specific pairs of graph classes $\mathcal{H}$, $\mathcal{G}$
  (Done for $\mathcal{G} = F$-colorable graphs, line graphs, claw-free graphs, ...)

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Thank you!

TikZ code for Kneser graphs available on GitHub

github.com/PH111P/tikz-kneser
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